Chapter XX

Learning Experiences Designed to Develop Multiplicative Reasoning: Using Models to Foster Learners' Understanding

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Successful progress in learning mathematics depends on a sound foundation of the understanding of multiplicative structures and reasoning. This includes not only the properties and meanings of multiplication and division, but also their many links with ratio and percentage and with rational numbers - both fractions and decimals. Yet these connections take time to establish and primary teaching can sometimes emphasise facility in calculation rather than the building of conceptual connections. This chapter draws on our experiences from the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) study in order to discuss how learning experiences can be planned to promote this kind of conceptual understanding. In doing so, the authors discuss the ways in which representations can be introduced and how formative assessment may be used.

1 Introduction

Multiplicative reasoning is a key competence for many areas of employment and everyday life (Hodgen & Marks, 2013) as well as for further mathematical study. It is however a complex conceptual field that cannot be reduced to a simple set of rules and procedures. Learners' early experiences, and understandings, of multiplication are likely to be dominated by repeated addition:

7×8 is conceived as 7 'lots of' 8 or 8+8+8+8+8+8+8.

Whilst repeated addition can be a powerful way of understanding multiplication, there is a great deal of evidence that understanding multiplication *only* as repeated addition can hinder learners' mathematical development and lead to errors and misconceptions (Anghileri, 2001). In particular, too great an emphasis on repeated addition may encourage learners to conceive of situations involving ratio or proportion as involving an additive rather than a multiplicative relationship. It is no surprise then that many researchers argue that sharing and equipartitioning provide a much stronger basis for the development of multiplicative reasoning (e.g., Confrey et al., 2009; Nunes & Bryant, 2009).

In this chapter, we begin by discussing some problems that children have in understanding multiplicative reasoning. We then consider the ways in which multiplicative reasoning can be understood and represented. Finally, we present – and discuss – two activities. In doing so, we draw on our work in the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) study.

2 Background

ICCAMS was a 4½ year project funded by the Economic and Social Research Council in the UK.[1] Phase 1 consisted of a survey of 11-14 years olds' understandings of algebra and multiplicative reasoning, and their attitudes to mathematics (Hodgen et al., 2010). Phase 2 was a collaborative research study with a group of teachers which aimed to improve learners' attainment and attitudes in these two areas (Brown, Hodgen, & Küchemann, 2012). Phase 3 involved a larger scale trial with a wider group of teachers and learners.

The Phase 1 ICCAMS survey involved tests of decimals and ratio first used in 1976 and 1977 in the seminal Concepts in Secondary Mathematics and Science (CSMS) study (Hart & Johnson, 1983). In 2008 and 2009, the decimals and ratio tests were administered together with an algebra test to a sample of around 8000 learners aged 12-14 from schools randomly chosen to represent learners in England.

The CSMS tests were carefully designed over the 5-year project starting with diagnostic interviews.[2] [3] The Decimals test focuses on decimals as an aspect of rational number and principally assesses two aspects of decimal number: ‘measurement’, and the ‘multiplicative’ areas of quotient and operator (Brown, Küchemann, & Hodgen, 2011). The Ratio test focuses on the use of ratio in a variety of situations with a particular focus on whether learners used multiplicative rather than additive approaches (Hart, 1980). One finding of Hart’s was that there is an intermediate strategy (‘rated addition’) for solving ratio problems, in which, for example, instead of multiplying a number by 2.5 a learner would first double and then halve the number, and add the results together. Although both operations of ‘doubling’ and ‘halving’ have a multiplicative basis, many learners do not initially appreciate this and see them as additive operations.

In Phases 2 and 3, we developed a set of *design principles* for which there is research evidence to indicate they are effective in raising attainment (Brown, Hodgen, & Küchemann, 2012). These included *connectionist teaching* (e.g., Askew et al., 1997; Swan, 2006), *formative assessment* (e.g., Black & Wiliam, 1998), *collaborative work* (e.g., Slavin, Lake, & Groff, 2009; Hattie, 2009) and the use of *multiple representations* (e.g., Streefland, 1993; Gravemeijer, 1999; Swan, 2008). The focus of this chapter is on the last of these.

3 What is Multiplicative Reasoning?

Our approach to teaching and learning multiplicative relations uses different ways of thinking about and representing situations which involve:

- multiplication
- division
- scale factors and rates
- ratio and proportion.

These situations may involve whole numbers used to quantify discrete variables, including very large whole numbers, and positive and negative integers. They also may involve positive and negative numbers used to measure continuous variables in measurement contexts, whether rational numbers (those expressible as fractions, or as either terminating or recurring decimals) or real numbers (including non-recurring decimals like $\sqrt{2}$ or π).

The focus is on developing learners' understanding of the meaning and structure of the relationship whose general form is $c = a \times b$, and to distinguish between additive and multiplicative relationships. A solid understanding of multiplicative reasoning underlies algebra and is at the heart of the functional relation $y = kx$ (see, e.g., Davis & Renert, 2009).

4 Learners' Difficulties with Multiplicative Reasoning

Learners have a great deal of difficulty with multiplicative reasoning. So, for example, one item on the Decimals test asks learners to choose which of the two operations, multiplication (×) or division (÷), produces the 'bigger' answer for operations involving related whole numbers and decimals (see Figure 1).

Ring the one in each pair which gives the **bigger** answer:

(a)	8×4	or	$8 \div 4$
(b)	8×0.4	or	$8 \div 0.4$
(c)	0.8×0.4	or	$0.8 \div 0.4$

Figure 1. The ‘which gives the bigger answer: × or ÷’ item from the Decimals test

This item is addressed at the common misconception, often referred to as *Multiplication Makes Bigger, Division Makes Smaller*, where learners incorrectly generalise from multiplicative contexts involving whole numbers (Greer, 1994). In 2008/9, only 19% of 14 year olds in England got this item completely correct.

Understandably, the focus of a great deal of teaching is to ensure that learners know *how* to carry out multiplication and division. Yet, ensuring that learners know *when* to use multiplication or division (as well as when not to) is equally, if not more, important. Several items on the Decimals test addressed this issue. For example, learners were asked which calculation should be used to work out the price of one litre of petrol given the cost of 6.22 litres (See Figure 2).

The cost of 6.22 litres of petrol
was £4.86.
What would the price of one litre be?

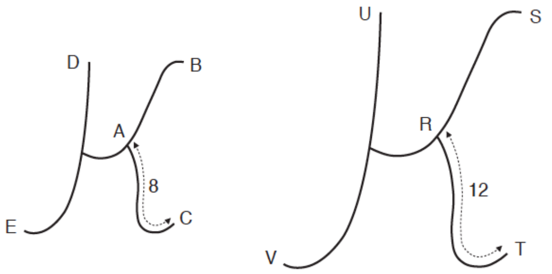
$6.22 + 4.86$	$4.86 \div 6.22$
$6.22 \div 4.86$	$4.86 - 6.22$
$6.22 - 4.86$	4.86×6.22

Figure 2. A ‘which calculation’ item from the Decimals test

In England, in 2008/9, only 17% of 14 years olds got this item correct. Many chose the incorrect response, 6.22 ÷ 4.84. This may be partly because they judged the division should always involve the division of a larger number by a smaller number, reflecting the *Multiplication Makes Bigger, Division Makes Smaller* misconception referred to above.

Contexts involving enlargement cause even greater difficulties. Item 7 of Ratio test (see Figure 3), often referred to as "Kurly K", involves relatively 'simple' multipliers [×1.5,× 2.25,× 2/3]. However, these are set in a 2D context that is much less amenable to a 'rated addition' strategy. So, whilst the numbers involved are relatively 'simple', the context is not. (For a discussion of these and related issues, see Küchemann, Hodgen, & Brown, 2010.)

These two letters are the same shape. One is larger than the other.
AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS ? . . . .

The curve UV is 18 units. How long is the curve DE ? . . . .

Figure 3. The "Kurly K" item from the Ratio test

In 2008/9, 13% and 15% respectively of 14 year olds got the first and second parts of this item correct. The following interview conducted with a group of English high attaining learners by two of the authors (Dietmar and Jeremy) illustrates learners' difficulties:

[The interview begins with a long silence and some whispering]

Bethan: 8 is kinda equal to 12, in the same way that 9 is equal to that [RS]

[Pause]

Dietmar: What's your next step?

Bethan: I'm still thinking ...

Zack: I don't really get it ... I was thinking ... if the 8 here [points at RS] and that's 9 [AB], you plus it to find out that? But ... is this 13? [RS]

Bethan: I'm still not sure ...

Jeremy: Why 13?

Zack: Difference is 4 ... [it's] larger by 4 ... so this [RS] should be larger (than the 9) by 4 ... if it's an accurate enlargement

Bethan: I think it's 13 and a half... cos 8 up to 12 is two-thirds... so 9 is two-thirds ... so I halved 9 which is 4.5 then add it onto 9 so you get

Jeremy: You think 13, you think 13.5 ...

Bethan: You halve it, then times it by 3 to get 3 thirds

Zack: Like 8 and 12 ... the difference 4, hey ... if you enlarge the smaller shape to get the bigger shape, wouldn't everything have to have the same ... difference?

Bethan: I'm not sure if you have to add something or times it by something ...

This, together with the item facilities, indicates that even higher attainers tend to use an addition strategy for enlargement. It is only a few higher attaining learners, like Bethan, who are able, even if tentatively, to see their way through to multiplication strategies, even when easy scale factors as here allow 'building up' or rated addition strategies. In fact in Bethan's two explanations it is possible to see a move from rated addition ('I halved… then add it onto…') towards multiplication ('Halve it, then times it by 3').

In the next section, we consider the complexity of multiplicative reasoning by examining the different models and representations that can be used.

5 Models of, and for, Understanding Multiplicative Reasoning

Different interpretations of the *meanings* of multiplication are described in the sections below. They can be described, or represented, using models. These involve pictures and diagrams, symbols, and text to provide tools which are usable to interpret and solve problems both in the real world and in a purely mathematical context.

It is worth emphasising that none of the models described below provides a complete understanding of the nature of multiplication, or a justification for the structures of multiplication. A connected understanding of multiplicative relations involves the ability to use a range of models like these, whichever is appropriate, with any type of number, and therefore takes a great deal of time to develop. But using the models does provide opportunities for learners to discuss and share their ideas, and for teachers to highlight structural aspects of multiplication.

The models are of course not always as distinct as might be suggested, and indeed different researchers have come up with different way of categorising them. For example, Davis & Renert (2009) worked with a group of teachers who produced a list of thirteen 'realisations of multiplication' which included transformations and stretching / compressing a number line. (See also, Lamon, 2005.)

Anghileri and Johnson (1992) identified six key 'aspects' of multiplication: repeated addition or grouping, arrays and areas, scale factors and enlargements, ratio and proportion, rates, and the Cartesian product. In the following discussion, we focus on just three of these 'models'. First, we consider the strengths and weaknesses of repeated addition and the grouping model. Then, we discuss arrays and areas and scale factors and enlargements. Elsewhere we discuss the use of the double number line, and the related aspects of ratio and proportion, and rates (Küchemann, Hodgen, & Brown, 2011; 2014).

5.1 *Repeated addition and grouping*

Generally, in school and beyond, most people think of multiplication in terms of repeated addition or in terms of grouping. Thus, 7×4 can be thought of as 7 *lots of* 4 or 7(4) or as $4 + 4 \ldots + 4$. Actually there is an ambiguity about the way the expression 7×4 is interpreted in English as it can be read as (7 times) 4 i.e. 7(4), but it can also be read as 7 (times 4), or 7 multiplied by 4, which would be 4 lots of 7 or 4(7). Taking it for the moment as 7(4), this can be represented using set diagrams showing e.g. 7 packs of 4 apples (Figure 4), or as jumps on a number line (Figure 5).

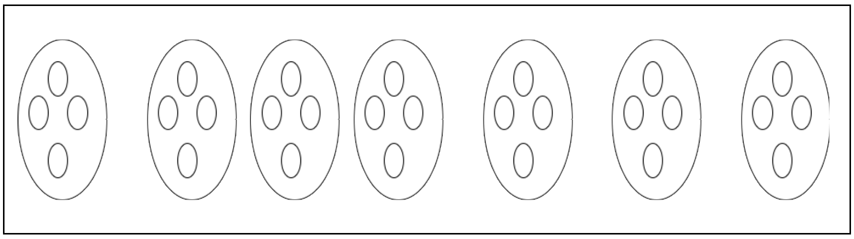


Figure 4. 7×4 modeled as 7 lots of 4

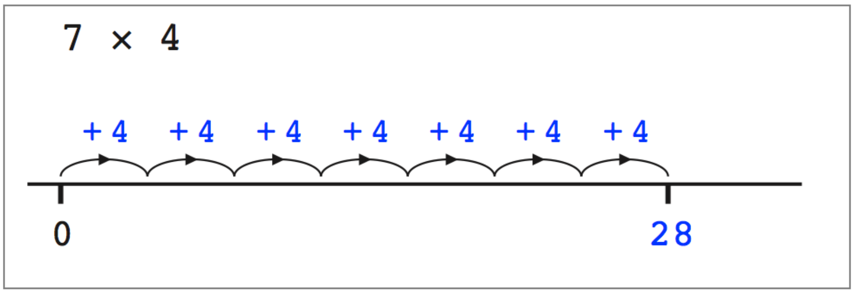


Figure 5. 7×4 modeled as 7 jumps of 4 along a number line

Generalising to the continuous case, 7 × 0.4 (e.g. for a problem about 7 glasses of 0.4 litres) can be thought of as 7 *lots of* 0.4 or as 0.4 + 0.4 … + 0.4 or as 7(0.4). This can be shown on a number line (as can 7 × -4) but neither can be illustrated by a set diagram. However repeated addition can't easily model a product of decimals like 0.4 x7 (unless reversed) or 3.7 × 0.4 or integers like -7 × -4

Division related to the repeated addition model is more complicated than multiplication, because any given division like 28 ÷ 4 can be thought of in two ways, as involving grouping (quotition) ("How many packs of 4 apples can be made from 28 apples?") or sharing (partition)

("How many apples in each pack if 28 apples are split into 7 packs?"). Similar differences arise using the number line model between grouping ("How many steps of length 4 make 28?") and sharing ("How long are the steps if 7 steps make 28?"). Solving the sharing problem, *If I pour 7 glasses from a 2.8l bottle, how much in each glass?*, is difficult using this model, as it has to be done by trial and error.

So, whilst repeated addition/subtraction is a useful and necessary model of multiplication / division, it has a number of limitations:

- It does not generalise to cases where both terms are rational or real numbers, or negative integers.
- It does not help learners think multiplicatively where a scaling idea is needed to solve enlargement problems, e.g. the *K* enlargement on the ICCAMS tests (perhaps because one can't sensibly add a *K* to another *K* to make a larger *K*).
- It encourages the misconception referred to earlier as Multiplication Makes Bigger, Division Makes Smaller, because learners incorrectly generalize from addition (i.e. from the fact that, for positive numbers, addition always results in an answer that is bigger than the original number).
- It is not at all easy to represent ratio and proportion problems using repeated addition.
- Similarly, considering algebra, whilst multiplicative relationships involving one variable, such as $4b$, can be straightforwardly represented on a number line, representing ab is more tricky as it involves a degree of imagination.
- Commutativity can be illustrated on a number line, but only to confirm that it is true: the number line does not really make clear why commutativity is **always** true.

5.2 *Arrays and areas*

Arrays (grids) and areas are very powerful tools for understanding the structure of multiplication. Whereas repeated addition represented multiplication as a unary model (one number being operated on by another), arrays and areas model multiplication as a binary operation, with the two numbers having symmetric roles. Thus 7 × 4 can be represented as a 7 by 4 array or a 4 by 7 array. The fact that the number of items in each of these are clearly equal provides a way of thinking about the commutative law $a \times b = b \times a$.

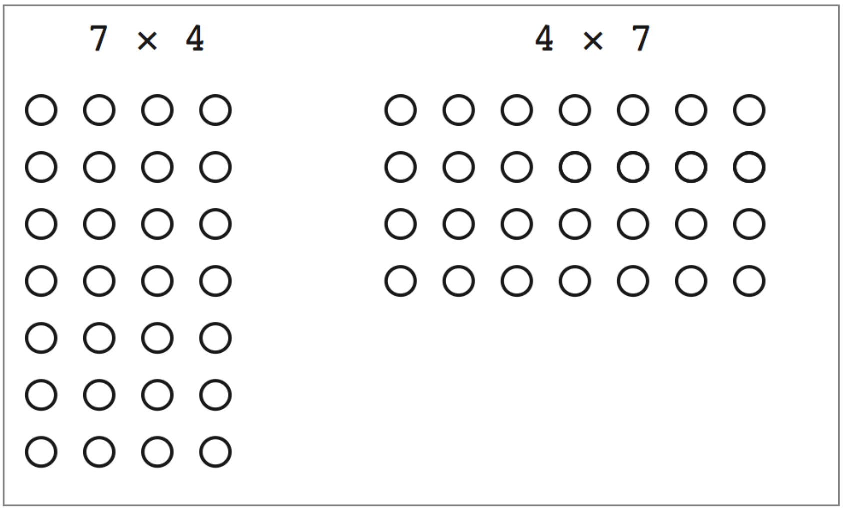


Figure 6. 7×4 and 4×7 modeled using arrays

Arrays can also be a useful way to explore factors and the associative law by discussing activities such as how many different arrays can be made from 24 dots.

The array can also be used to think about the distributive law and partitioning. For example,

$$17 \times 24 = (10 + 7)\ \ (20 + 4)$$
$$= (10 \times 20) + (7 \times 20) + (10 \times 4) + (7 \times 4)$$

Whereas arrays involve discrete contexts, areas use what is virtually the same model but with continuous contexts and numbers expressed as decimals or fractions.

For division problems, arrays and areas can be very useful in helping learners to understand division as the inverse of multiplication, leaving behind ideas of repeated addition and subtraction. Hence, we can solve 72 ÷ 12 by asking what number do you multiply 12 by to get 72. Similarly, we can model 7.2 ÷ 0.6 using a rectangle of side 0.6 and area 7.2.

It is important to note that areas are not absolutely straightforward as learners may not fully understand what area is (i.e. finding the number of unit squares that cover a region completely). It is likely that they will be developing their understandings of area and multiplication alongside each other. So, on the ICCAMS Algebra test (see Chapter XX of this book), when asked the area of a 5 by e+2 rectangle, only 13% of 14 year olds in England answered correctly in 2008/9 (see Figure 7).

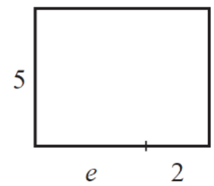


Figure 7. A 'What is the area of this shape' [5 × (e+2)] item from the Algebra test

Areas also generalise to algebra in the use of areas as models for the multiplication of variables. The textbooks tend to assume that this representation is transparent. But unless learners have some experience of using areas to represent multiplication, the multiplicative relationships that are at the heart of the algebra are not at all clear. Hence, learners may be learning *how* to work out problems involving symbols and shapes rather than using area as a tool / representation of multiplication to develop their understanding of algebra.

As with arrays, an important feature of areas is that the commutative law is powerfully represented, and the symmetry of the two numbers or variables involved. This in turn can help learners to abstract from particular models and contexts by developing an understanding of how they can use what works best for the numbers (and to develop algebraic thinking).

5.3 *Scale factors and enlargements*

The idea of a multiplying factor is really a generalisation of the repeated addition model where 7(4) could be thought of as 7 lots of 4 (e.g. 7 packs of 4 apples), and where 7 could therefore really be thought of as a multiplying factor operating on the number 4, with ×7 as a unary operation.

Although the repeated addition model required the multiplying factor to be a whole number, the extension of the notion of a multiplying factor to a scale factor acting on a length on the number line allows a generalisation to a rational number as multiplier. Thus 3.7×0.4 can be thought of as enlarging (stretching) a number line by a scale factor of 3.7, so that a length of 0.4 becomes $3.7 \times 0.4 = 1.48$. Similarly 0.4×3.7 becomes an 'enlargement' of the number line by a scale factor of 0.4, which is less than 1 and hence a shrinking.

We can symbolise this enlargement relationship as $y = m \times x$, since the scale factor, m, is constant and the initial length on the number line, x, varies. This is the case therefore that has the closest relationship with the linear relationship $y = mx + c$. It also means that it makes sense to represent an enlargement on a graph as $y = mx$.

In 2 dimensions we could 'stretch' each dimension by a different scale factor but in enlargement problems relating to similar figures we have a single scale factor, as in the *K* enlargement, so we are interested in learners understanding that the scale factor is constant for all corresponding elements of each shape. So, each length, x, in the smaller *K* is enlarged by a scale factor of $^3/_2$, thus becoming $(^3/_2 \times x)$ in the enlarged *K*. Again the relation can be graphed as $y=3x/2$ or $y=1.5x$.

There are two forms of division relating to scaling problems, one of which reflects grouping and one sharing. For the former, we can use two corresponding lengths to find out the scale factor, which corresponds to dividing 28 apples by 4 apples to find the multiplying factor of 7, which was a grouping problem. While it is in the discrete case solvable by repeated addition or subtraction, in the context of enlargement this doesn't always make any sense (e.g. how many small curly sides can you put together to make a curly side 3.7 times as long?).

The 'sharing' model of division ($28 \div 7 = 4$) corresponds to the other enlargement division problem where the resulting length is known and the scale factor, and we have to work out what was the length we started with.

6 Using Models to Develop Multiplicative Reasoning

In this section we will briefly describe tasks used in ICCAMS lessons aimed at developing models for multiplication and ratio.[4]

6.1 Whole number multiplication models

Since one focus of the design of ICCAMS teaching was on formative assessment (Hodgen & Wiliam, 2006), an early lesson in the sequence is intended to allow the teacher to assess where learners are starting from in their understanding of the different verbal and visual models for multiplication.

Models and stories

Here is an expression involving 12 and 3:

$$12 \times 3$$

Think of
a. some ways of saying "12 × 3"
b. some ways of calculating 12 × 3
c. some diagrams that fit the expression
d. some stories that fit the expression.

Figure 8. An assessment task: Models and stories for 12 × 3.

The task in Figure 8 is designed to elicit a range of models for multiplication and, thus, to enable class discussion of different models proposed by learners, as well as suggesting to the teacher how much work is needed to build familiarity with the models before proceeding to the introduction of more sophisticated work on scale factors and ratio.

6.2 *Using areas to model multiplication with rational numbers*

A key issue is to extend the area model from multiplication with whole numbers to multiplication with rational numbers. The task illustrated in Figure 9, begins by considering 3.2 × 2.6 as 3.2 rows of 2.6 unit squares. Learners can estimate the number of squares as well as discuss how many unit squares there are in the "thin" column to the right, the "short" row at the bottom, or the small rectangle on the right at the bottom.

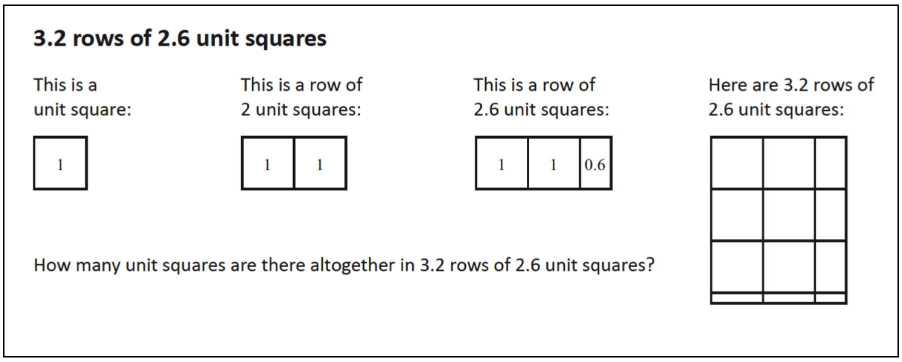


Figure 9. The task involving 3.2 × 2.6 as 3.2 rows of 2.6 unit squares

This meaning is a little strained for fractional numbers. Hence, the lesson moves on to a task (involving 5.3 × 2.6) in which the area is thought of as distances along orthogonal number lines whose product is an area (whilst the area is still conceived of in terms of unit squares).

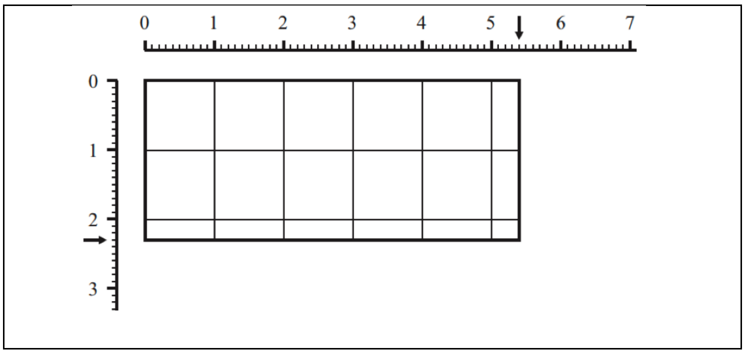


Figure 10. The task involving 5.4 × 2.3 using orthogonal number lines

6.2 *Modeling relationships involving scaling*

Here we present part of a sequence of related scaling tasks set in the context of shadows, where the tasks are used to diagnose learners' understandings in order to adapt the task adapted to learners' needs. The numbers in Figure 11 have deliberately been chosen so that the (functional) relationship between the length of a post and its shadow involves a fractional multiplier (in this case ×2 1/2) rather than a whole number multiplier, and so that the same applies to the (scalar) relationship (in this case ×2 1/6) between the lengths of the two poles (and between the lengths of their shadows). The fact that the context involves ratio may be far from obvious to some students.

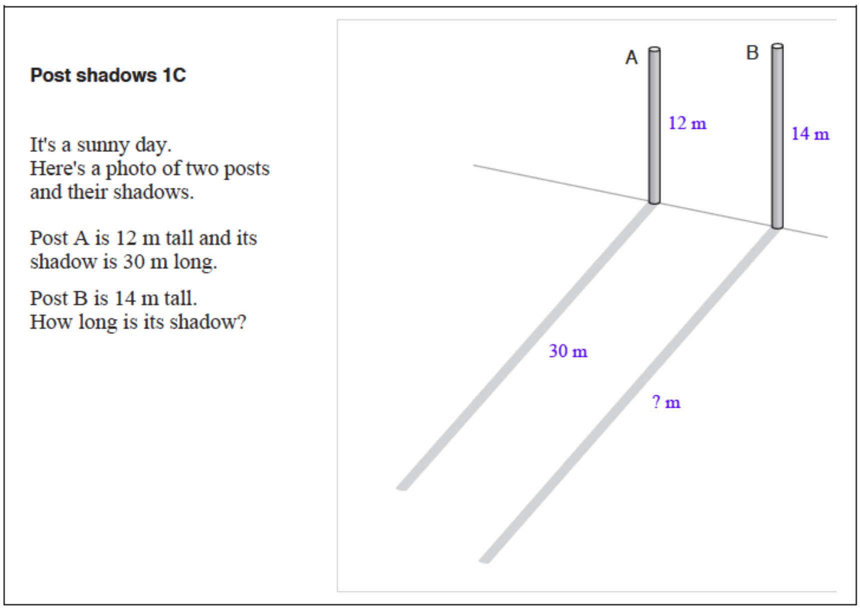


Figure 11. The initial 'Shadows' scaling task[5]

It is likely that these relatively difficult numbers will prompt some students to adopt what has become known as the "addition strategy" and hence conclude that the length of B's shadow is 32 m rather than 35 m (derived from 14 m + 18 m, or 30 m + 2 m). If so, learners are presented with the task in Figure 12. Here, post B has been partitioned into two sections, one of which is the same length as post A and which hence has a shadow of 30 m. The other section is just 2 m long. The task now is to find the length of the shadow of the 2 m section. The intention here is to provoke a conflict for those students who attempt to apply the addition strategy again, as this leads to a shadow length of 20 m (12 m + 18 m, or 30 m – 10 m), which is far larger than the actual length of 5 m.

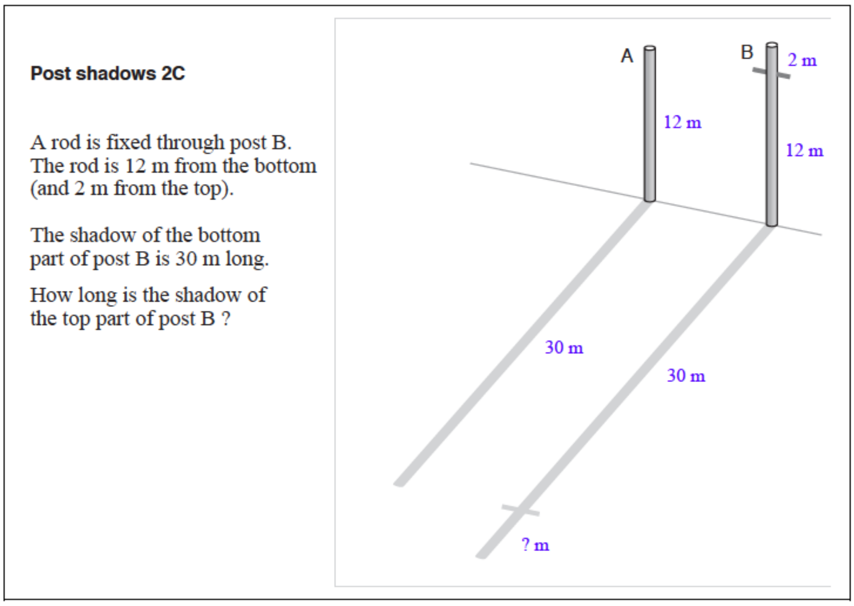


Figure 12. The 'Shadows' scaling task adapted for learners using the addition strategy

If most learners come up with the correct answer of 35 m on Figure 11 (and thus don't use the addition strategy), they may be presented with the task in Figure 13. In this task the sun has moved to a slightly lower elevation, causing the shadow of the 12 m post to lengthen by 1 m. The (functional) relationship between post length and shadow length is now numerically even more complex (×2 7/12 rather than ×2 1/2), which may lead some students to resort to the addition strategy and hence give an answer of 36 m (35 m + 1 m) or 33 m (31 m + 2 m, or 14 m + 19 m) for the new shadow of post B, rather than 36 1/6 m.

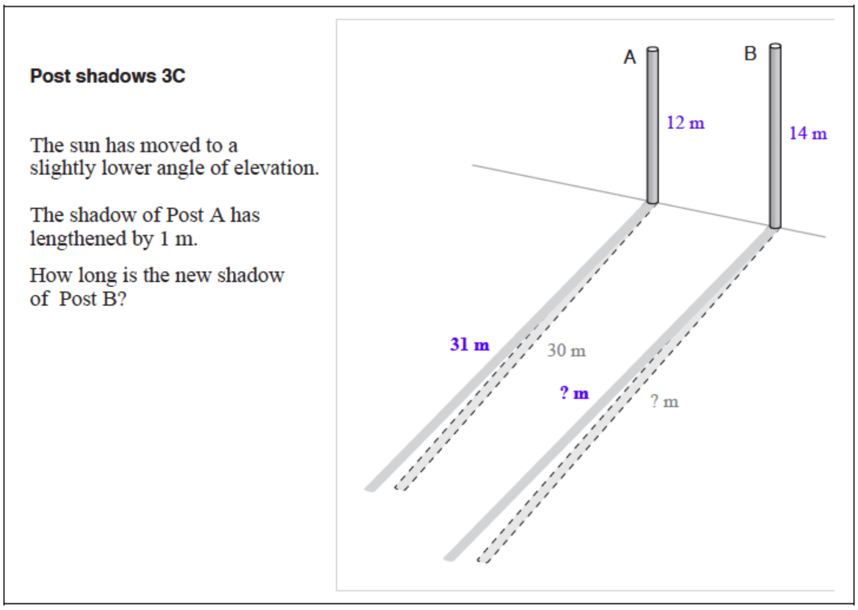


Figure 13. The 'Shadows' scaling task adapted for those using

On the other hand, those learners who have a good understanding of the multiplicative relationship between the length of a post and its shadow may have no markedly greater difficulty in deciding how to find the relationship (e.g. divide 31 by 12) and how to apply it to find the new shadow of post B (e.g., 14 × (31 ÷ 12)) - though some may need/want to use a calculator to carry this out.

7 Conclusion

The ICCAMS lessons and approach was designed and trialled in the English context. In this context, the wider trial in Phase 3 of the ICCAMS study showed a significant effect: over a year the rate of learning for those who had experienced the lessons was double that of those who had not (Hodgen et al., 2014). We have focused here on three models for developing understanding of multiplicative reasoning and shown how the repeated addition model, whilst important, is not sufficient for developing multiplicative reasoning as it breaks down for multiplicative situations such as scaling, and when we shift from the natural numbers to integers and rational numbers

Notes

1. ICCAMS was part of the Targeted Initiative on Science and Mathematics Education (TISME) programme. For further information, see: tisme-scienceandmaths.org/
2. The Decimals and Ratio tests are available for non-commercial purposes (research and teaching) by contacting the authors.
3. A few additional items from the original Fractions test were added to the Ratio test in order to assess learners' understanding in this area. Piloting indicated that only minor updating of language and contexts was required for the 2008/9 administration.
4. The ICCAMS lessons are available for trialling by interested teachers and schools by contacting:
jeremy.hodgen@kcl.ac.uk or dietmar.kuchemann@kcl.ac.uk.
See also: http://iccams-maths.org
5. Note that this picture is drawn in the Northern hemisphere.

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