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**CHANGING THE GRADE 7 CURRICULUM IN ALGEBRA AND  
MULTIPLICATIVE THINKING AT CLASSROOM LEVEL IN RESPONSE TO  
ASSESSMENT DATA**

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This paper reports one part of the Increasing Confidence and Competence in Algebra and Multiplicative Structures (ICCAMS) project. The results of Phase 1, a survey of attitude and understanding in the areas of algebra and multiplicative thinking of students in grades 6-8, revealed specific areas of difficulty. Some of these are illustrated by test item results. Phase 2 involved working with teacher researchers and their grade 7 classes to explore ways of improving understanding in these areas of difficulty. Phase 3 of the project entailed the trialling of the resulting teaching activities in 15 schools. The activities incorporated formative assessment and rich tasks, involving multiple representations. Examples are given which relate to the described areas of weakness in the survey. Teachers say they have changed their ways of teaching and hopefully our results will also demonstrate that they have improved student understanding of these topics. Algebra; multiplicative thinking; formative assessment; rich tasks; multiple representations.

## **INTRODUCTION**

The project described in this paper was one of a series funded in the UK by the Economic and Social Research Council as part of a themed programme of research aimed at widening participation in STEM subjects (Science, Technology, Engineering, Mathematics) in the later years of secondary school and university. Our research team at King's College London, having considered the existing research on participation in mathematics (e.g. Matthews and Pepper, 2007; Brown, Brown and Bibby, 2008), felt that the main obstacles to participation lay in negative student attitudes; most students did not want to carry on with their mathematical studies because they believed they were not 'good at mathematics', and 'did not understand it'. They also found it 'boring' and 'unrelated to real life'.

Two mathematical areas which are a key part of the age 11-14 curriculum but which seemed to cause particular problems to students were algebra, and multiplicative thinking (ratio, including the multiplicative use of rational numbers). Algebra, although not perceived as useful by most students and adults, is particularly important in relation to further study in mathematics and in subjects which draw heavily on mathematical modelling. Multiplicative thinking is central not only in mathematics but in the application of mathematics in employment and everyday life, especially using percentages and proportions.

Our project, which was accepted for funding for the period 2008-12, therefore had the title '*Increasing Confidence and Competence in Algebra and Multiplicative Thinking*

(*ICCAMS*)'. It aimed at increasing student participation through improving their understanding of these topics, and, through this, their confidence in their ability to do mathematics. Additionally it also aimed at demonstrating the importance and power of mathematics and its real-life applications.

The project had three phases:

*Phase 1* took the form of a survey of attitude and understanding in the areas of algebra and multiplicative thinking, in order to identify which ideas students found most challenging.

*Phase 2* involved the Research Team at King's working alongside 8 teacher researchers to interview their grade 7 students in groups about the areas of difficulty identified in Phase 1 and to try out new approaches which might help to build student understanding, both with groups and with whole classes.

*Phase 3* involved trialling this approach on a larger scale

In this paper some items with low results in Phase 1 will first be described and then related to activities used in Phase 3 to attempt to improve students' understanding and confidence. Although there is substantial evidence from interviews and observations in Phase 2 which provides more details about student understanding and the varied success of attempts made to deepen this, there is not much space to refer to this in this paper.

Some background is needed about the state of mathematics teaching in the UK during the project. Schools are subject to the publication of annual league tables of test results and frequent inspections. In 2001/2 a Secondary National Strategy was implemented in mathematics which proposed teaching objectives for each block of lessons in each grade; although not statutory almost all schools and textbooks followed these to guard against criticism by inspectors. Although the new government have abandoned these, during the project most schools still followed them.

The result of these features was that teachers were focused on teaching to the test and achieving narrow procedure-related objectives in each lesson (Ofsted, 2008). The curriculum therefore became fragmented with little clear rationale other than scoring well on short routine test items.

The aims of the project which involved increasing connected understanding, an attitude of confidence and to a lesser extent an appreciation of the power and applications of mathematics were thus in contrast to the culture prevailing in most schools. While some school managers, mathematics departments and teachers viewed the *ICCAMS* project positively as a way of improving results, the quality of formative assessment and pupil engagement to counter this culture, others were reluctant or experienced some opposition to trying out innovative ideas and practices due to heavy pressure on delivering success in high stakes tests. Some teachers found themselves short of time to attend meetings and master the new approaches when they were under pressure to spend many hours outside regular teaching timetables coaching other groups and individuals for the high stakes tests.

Last names of authors in order as on the paper

## PHASE 1

### Methods

*Test design* The survey used tests of Algebra, Ratio and Decimals developed in the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981). The Ratio test was supplemented by a small number of items from the CSMS Fractions test. Decimals and Fractions assessments were included as many items related to multiplicative thinking involving rational numbers. The CSMS tests were designed and trialled on the basis of results from diagnostic interviews, and the final versions were administered in 1976 and/or 1977. The focus was on conceptual understanding and application, although for completeness a very small number of items were designed to assess mathematical procedures.

A check was made that the items were still relevant to the current curriculum. Some minor updating changes were made and some additional items on algebra in the context of spreadsheets were provided (but not well answered!)

*Participants* Over two summers in 2008 and 2009, tests were administered to a sample of approximately 6000 students across grades 6,7 and 8 from 19 schools randomly selected within strata of performance on the MidYIS database which includes a large number of English schools (Tymms & Coe, 2003). This sample is approximately representative of the English population with a mean MidYIS ability score of 103.6 compared to a national average of 103. (This is the same method of selecting the sample as was used in the 1970s, except that a standardized non-verbal IQ test was used instead of the MidYIS test as a control.)

Each student took two of the three tests so as to provide comparative information between tests but not to overload students. The numbers of students in each year-group taking each test is therefore around 1300.

### Selected findings

In all the tests, results were the same or slightly worse than in the 1970s, with more children now scoring very few marks, and fewer scoring very high marks.

*Algebra* Over 70% of students were not consistently able to treat letters as specific unknown numbers or as generalised numbers or variables. They could succeed only in simple substitutions of numbers for letters or where letters could be regarded as objects. The proportion of students who could apply the idea of variable was less than 10%.

This is illustrated by the answers to item 16:

*If  $c + d = 10$  and  $c < d$ , what can you say about  $c$ ?*

The most common responses given across grades 6-8 on this item for the 1722 students is shown in Table 1 (the code numbers were used by markers).

Table 1: Relative frequencies of different types of response to item 16 on Algebra test

Response	Omit	Wrong	Single	Correct	C is less	C<5
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	or no real attempt	numerical answer or range	number (e.g. 4)	systematic set of integers(e.g. c=1,2,3,or 4)	than 5 (words)	(symbols)
Codes	0, 99	81-86	61-65	24	11N	11
Relative Frequency	44%	5%	36%	9%	5%	1%

The most common specific response was ‘4’ which was given by 29% of students, although 39% of students omitted the item completely. The proportion of students who did no more than substitute a single value was 85%, with the remaining 15% appreciating that the letters could take several different values and still satisfy the conditions.

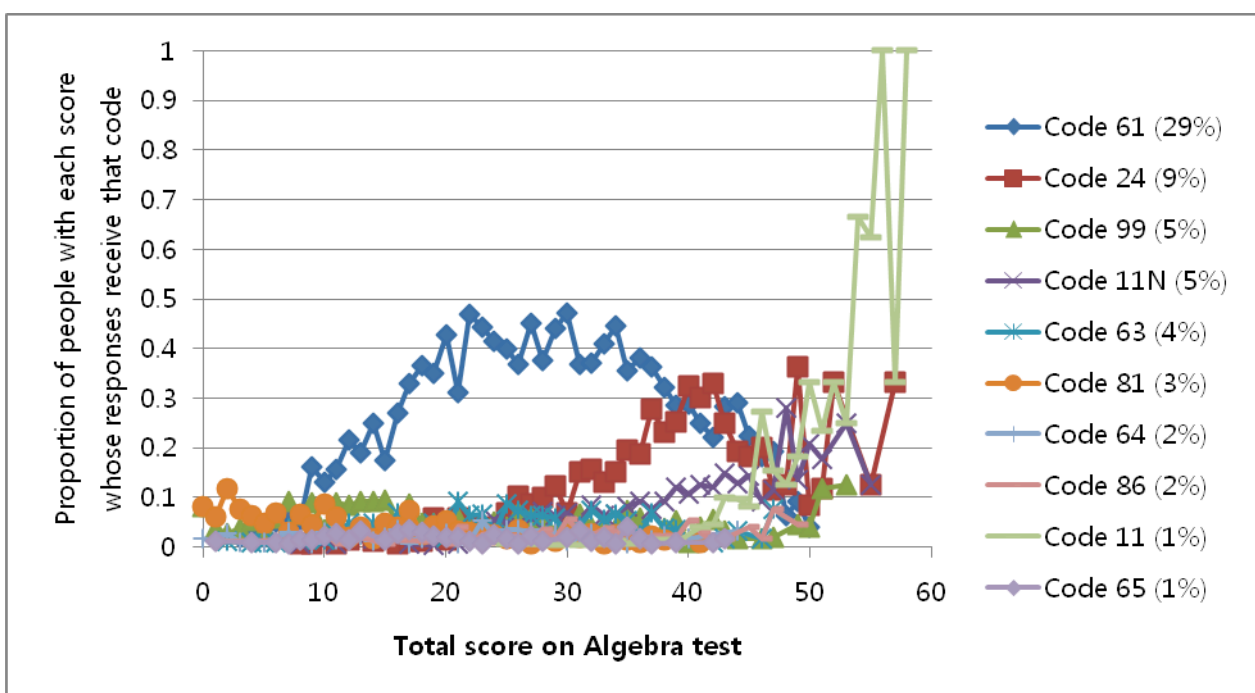


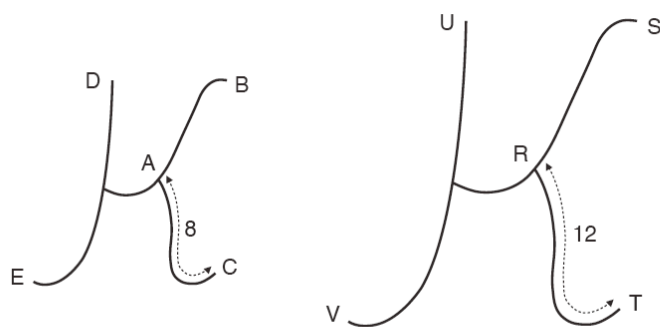
Figure 1: Frequencies of different answers to item 16 on algebra test by total score

Figure 1 shows how the different responses relate to the overall test scores (omits are not shown). This demonstrates the fact that the single number ‘4’ is mainly given by students with average test scores, while only about 6% of students with the highest test scores give a correct symbolic response.

*Ratio* Results in Ratio again were similar in showing that across grades 6-8, around 20% were not making any real response and in total more than 60% could not manage anything involving more complicated ratios than doubling, halving or multiplying by 3. About 30% could manage ratios which required building up from a given ratio or adding on half as much. Fewer than 20% could deal with more complex number and less than 5% could use ratio for scaling and enlargement. In particular as in the 1970s, there were a large number of students who used an addition strategy rather than a multiplication operation. An example where this is common is Item 7 in the Ratio test, shown in Figure 2.

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These two letters are the same shape. One is larger than the other.  
AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS ? .....

The curve UV is 18 units. How long is the curve DE ? .....

Figure 2. Item 7 in the Ratio test with parts 7a and 7b

In this item about 13% of students across grades 6-8 gave the correct answer of 13.5 to 7a and only slightly more gave the correct answer of 12 for 7b. More than 3 times as many students in gave the answer 13 to part 7a as gave the correct answer 13.5. The addition strategy is not the only way to obtain this answer (some students estimate) but in our interviews in Phase 2 many did. This example illustrates the different strategies of students E and S:

E: the difference is 4... larger by 4... so this (RS) should be larger [than the 9] by 4.. if it's an accurate enlargement

S: I'm not sure if you have to add something or times it by something... 8 to 12 is like two-thirds, .. so 9 is  $\frac{2}{3}$  of RS, so one-third is 4.5 cos you halve it, then times it by 3 to get 3 thirds...or a whole, which is RS which is 13.5.

### PHASES 2 AND 3

In Phase 2 different approaches were trialled to attempt to build a firmer understanding of algebra and multiplicative thinking. We used a large amount of research literature which informed us about developing thinking in multiplicative reasoning and algebra. There is not space to list all the references that proved useful sources of ideas, but they included on multiplicative thinking references such as Confrey at al., 2009; Harel & Confrey, 1994; and on algebra, references such as Sutherland et al., 2000; Mason et al, 2005.

Another basis of the work was to use generic pedagogical principles where there was research evidence that they are effective in raising attainment, in particular:

- formative assessment (Hodgen & Wiliam, 2006; Wiliam et al, 2004; Hattie, 2009)
- connectionist teaching (Askew et al, 1997; Swan, 2006)
- collaborative work (Slavin et al, 2009; Hattie, 2009) .

We also found it useful to build connections using multiple representations (Streefland, 1993; Gravemeijer, 1999; Swan, 2008).

The formative assessment element focused on providing both brief lesson starters and longer rich activities, in the form of non-routine questions which stimulate students' mathematical thinking and help to expose their ideas to teachers (and each other). This enabled teachers to have a better knowledge of the state of students' understanding, and to plan further activities appropriately. In particular we sometimes linked a brief starter which was left open to a later lesson which explored more deeply in order to allow time for teachers to reflect on answers and use them to better plan the later lesson. Generally students worked in collaborative pairs or groups to better develop and communicate ideas before explaining them to the class.

The rich tasks were non-routine problems designed to connect different mathematical ideas and representations and often to connect the mathematics to realistic contexts .

The teacher researchers who helped develop and trial the lessons alongside the university research team suggested that teachers would value most a set of lessons which were very briefly described for periods when teachers were very busy, but which in later sections provided greater detail and background for when teachers had more time to read and prepare.

The final form of the intervention is a collection of interlinked sequences of 40 outline lessons for Grade 7 students, 20 each on algebra and multiplicative reasoning. These are designed to allow engagement from a broad range of students, and to highlight opportunities for teacher adaptation, judgement and assessment.

The Appendix contains examples of pupil tasks and excerpts from teacher notes which illustrate the approach. The Boat Hire activity is intended to help students work towards an understanding of letters as representing variables to tackle the difficulty described earlier in the  $c+d=10$ ,  $c<d$  item. The Westgate Close item is aimed to assist students in adopting multiplicative rather than additive approaches, a problem identified in the 'Curly K's' item. Both these lessons involve the use of multiple representations of relationships.

In Phase 3 of the project a further 20 teachers from an additional 11 schools joined, trialling the lessons and contributing ideas for their adaptation. The approach was evaluated by comparing gains in achievement and attitude across the year with national norms established during the survey in Phase 1. Teachers were also interviewed about how they had implemented the approach and any difficulties they had experienced. We are currently analysing the results to see whether and how these changes have affected student learning in algebra and multiplicative thinking, and hope to be able to report some results at ICME. It is already clear from teacher interviews that the intervention has radically affected the teaching style of some teachers, and caused them to adopt ways of teaching which are in strong contrast to the current culture described earlier.

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**Appendix A: Draft Teachers’ Notes for Boathire Task**

**Boat Hire**

Olaf is spending the day at a lake.  
 He wants to hire a rowing boat for some of the time.  
 Freya’s Boat Hire charges £5 per hour.  
 Polly’s Boat Hire charges £10 plus £1 per hour.  
 Whose boat should Olaf choose?

**Summary**

In this lesson, the boat hire problem is used to explore the two algebraic relationships underlying Freya’s and Polly’s different hire charges.

A variety of representations are used to express the relationships:

- everyday language,
- algebraic expressions,
- tables of values,
- points on a Cartesian graph.

**Outline of the lesson**

Display the Boat Hire problem and ask students for their immediate responses.

Ask students to consider the problem further in small groups.

Collect numerical data on the total cost for various numbers of listen to students’ arguments and conclusions - but don’t pursue this stage).

Represent the data

- ‘randomly’ on the board
- in (randomly ordered) tables
- in ordered tables (try to prompt the need for this, rather than simply produce such tables).

Ask students to represent the hire-rules as algebraic expressions  $10+a$  or algebraic relations (eg  $b = 5a$ ,  $b = 10+a$ ).

Ask students to represent the data as points on a standard (Cartesian) graph.

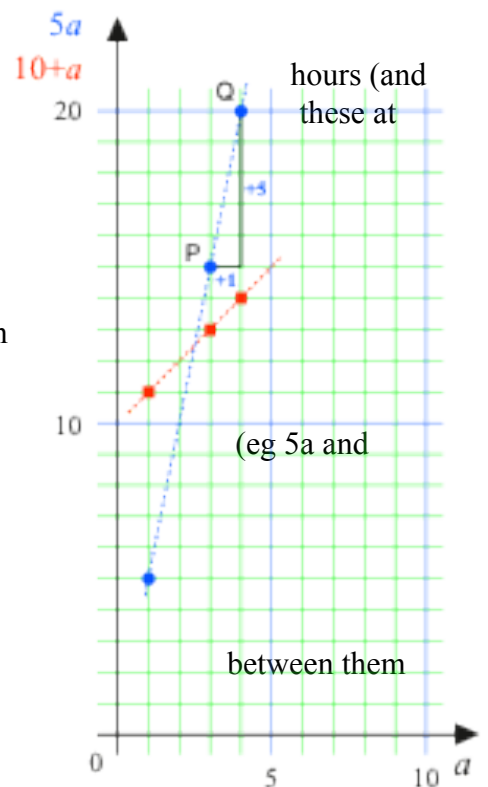
Discuss, use, make links between the various representations and and the story.

not ordered

$a$	$5a$
1	5
4	20
3	15

ordered

$a$	$5a$
1	5
3	15
4	20





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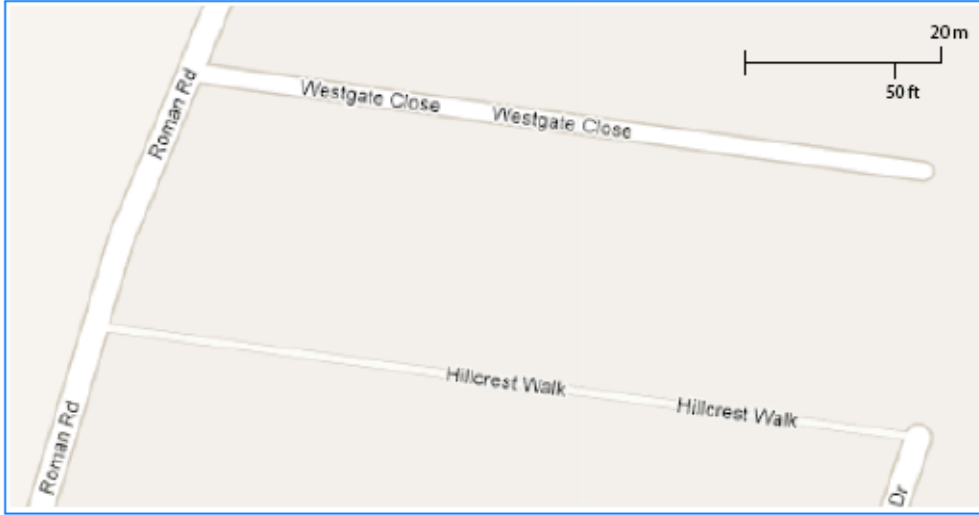
**Appendix B: Slides for Westgate Close Task**

**Westgate Close Task A**

Look at the scale on the map.

Use the scale to estimate the length of Westgate Close

- i. in metres
- ii. in feet.



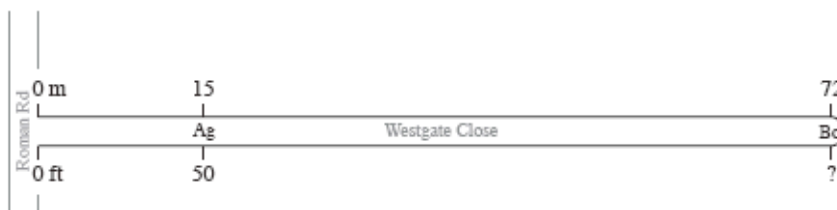
**Westgate Close Task B**

Here is another map of Westgate Close and Roman Rd.

Ag's house is 15 m, or 50 ft, along Westgate Close.

Bo's house is 72 m along Westgate Close, at the very end of the road.

- i. Estimate Bo's distance in feet.
- ii. Calculate Bo's distance in feet (try to find several ways of calculating the answer).



## Appendix C: Teachers' Notes for Westgate Close Revisited Lesson

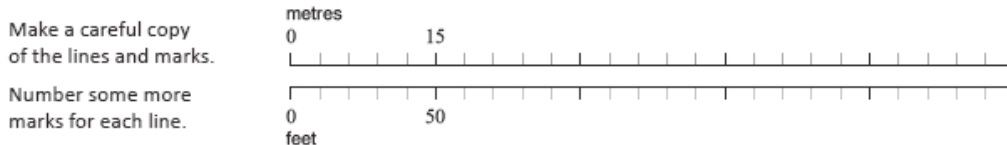
Multiplicative Reasoning: Lesson 1B

### Westgate Close Revisited

These two lines show distances (in metres and feet) along Westgate Close. In particular they show the distance (15 m or 50 ft) along Westgate Close of Ag's house in Task B.



The marks on the lines are in 5 mm intervals.



### Summary

In this lesson, students consolidate ideas from Lesson 1A by drawing and carefully numbering a double number line to serve as a device for converting metres to feet (and feet to metres). We then create similar devices using a mapping diagram and a Cartesian graph.

### Outline of the lesson

- Students draw and number a double number line to represent metres and feet.
  - Show the two partially numbered lines and remind students of Task B.
  - Ask students to carefully copy the lines and to number some (or all) of the marks. Let students decide which marks to number.
  - Discuss the marks - what numbers do various marks represent?
- Use the numbered double number line.
  - Locate the positions of the other numbers from Tasks B and C.
  - Discuss some other conversions (of your or the students' choosing).
- Draw a mapping diagram for metres and feet.
  - Draw parallel number lines, but this time with the same scale.
  - Draw arrows to represent (some of) the data from Tasks B and C.
  - Discuss some other conversions and compare the two diagrams.
- Draw a Cartesian graph for metres and feet.
  - Draw Cartesian axes with the same scale as for the mapping diagram.
  - Represent (some of) the data from Tasks B and C on the graph.
  - Discuss some other conversions and compare the three diagrams.

