The struggle to achieve multiplicative reasoning 11-14

Margaret Brown, Dietmar Küchemann and Jeremy Hodgen

King's College London

Multiplicative reasoning is a key competence for many areas of employment and everyday life, and for further mathematical study. It is however a complex conceptual field. The ICCAMS project, with multiplicative reasoning as one of its two focuses, has in Phase 1 conducted a broadly representative survey of attainment which suggest that standards in this area have not risen since the 1970s and that relatively few students are achieving competence in the relevant areas of the national strategy Key Stage 3 framework. Student difficulties are illustrated by evidence from group interviews in Phase 2 of the project.

Keywords: Multiplicative reasoning, ratio, assessment, key stage 3

Background: multiplicative reasoning

The ESRC project 'Improving Competence and Confidence in Algebra and Multiplicative Structures' (ICCAMS) selected these two areas of the curriculum because of their pivotal role in the Key Stage 3 (age 11-14) mathematics curriculum, and especially for the further study of mathematics and for its functional application.

While algebra centrally underpins the whole of the further study of mathematics, it is the basis for only the more sophisticated applications. In contrast multiplicative reasoning is the basis of most mathematical applications and is relevant to all pupils. This paper will examine aspects of performance in multiplicative reasoning that students find difficult and which form a major barrier to developing competence and confidence in functional applications.

The delineation of the conceptual field of multiplicative reasoning is complex (Harel and Confrey 1994, Confrey et al. 2009). While there are other aspects of multiplicative reasoning e.g. those concerned with combinatorics or calculation of areas and volumes, applications of the ratio/rate model are by far the most common and only these will be discussed in this paper.

Broadly speaking, the main contexts of ratio/rate application are those where two or more values are being compared, and/or where one value is being scaled up or down to give another. Sometimes these comparisons or operations are appropriately expressed additively and involve the ideas of difference ('a is d more/less than b'); more often they are multiplicative ('a is r times bigger/smaller than b').

The values being compared may be of essentially the same quantity and the comparison may be related to two or more different 'things', or to one 'thing' at two or more different times (numbers of boys and girls; changing numbers of boys). Especially where the quantity is a discrete variable, or where the multiplying factor is a whole number or familiar fraction, this relationship is often expressed as a ratio a:b. Where the comparison is between two variables which refer to different quantities measured in different units, the relation is usually expressed as a rate a/b which takes the form of a single number to which is attached a composite unit like miles per hour or £ per capita derived from the units in which the two variables are measured. These comparisons are equivalent in reverse to scaling up a value to get another value, respectively by using a ratio or scale factor, or by using a rate.

Whereas a specific value of a rate is a single number with a composite unit, a specific ratio a:b can be regarded as a set of pairs of numbers which is associated with two 'dimensionless' rates a/b and b/a. Such a dimensionless rate is often described as a proportion, especially when it is expressed as a fraction or percentage and refers to a comparison between one contributing part and a whole collection ('what proportion of the class are boys?'), or to similar geometrical figures ('have the same proportions') . 'Direct proportion' is also used more generally to describe a multiplicative relation between two variables.

Rates often but not always involve the variable of time and are very commonly used in finance and economics (e.g. GNP per capita, rates of interest, and exchange), in the physical sciences (e.g. speed, density, power, pressure) and in health (e.g. rates of growth, medicine doses). Other applications use dimensionless rates (proportions), for example probability and risk, and enlargement through scaling in spatially focused professions such as architecture, design, and engineering.

Thus all the ratio/rate applications have in common the two processes which constitute multiplicative thinking: the derivation of a rate (or ratio or proportion) from two corresponding values (a/b) of two or one variables, and the use of a rate (or ratio or proportion) to calculate an unknown value of one variable given a corresponding value. These require respectively the operations of division and multiplication, with division if anything taking the predominant role.

In our primary curriculum the aspect of multiplication which is still most emphasised is that of repeated addition ('add three five times') rather than that leading to multiplicative reasoning and ratio ('five for every one of three', 'five times larger than 3'). This is even when the ratio meanings have been demonstrated to be easier (Nunes and Bryant 2009).

Multiplicative reasoning starts in the primary school with whole number quantities and whole number scale factors. Later in primary schools and especially in Key Stage 3, it becomes tied in strongly with rational number reasoning. Dickson et al (1984, 287) note there are two meanings of rational numbers, those related to measurement (1.7 metres) and those to operators – essentially rates or scale (multiplying) factors (1.7 times as large); Confrey et al. (2009) separate off from the latter a further meaning, that of ratio, but there seems little justification for doing so since ratios are so closely related to the operators of rates and scale factors.

Again, the non-ratio 'measurement' meanings of rational numbers as a measure of a fractional part of a spatial whole, or as a decimal number on a number line, are the ones that tend to be emphasised in the primary curriculum, although there is some mention of meanings concerned with ratio ('3 for every 5', '3/5 of').

Perhaps not surprisingly, given the complexity of the conceptual field on which it depends, and current emphases in primary schools, in the early years of secondary school students experience problems with multiplicative reasoning. This paper explores some of these using early results from the ICCAMS project.

Methods: the ICCAMS project

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics (STEM) disciplines.

Phase 1 of the project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England. This is followed in Phase 2 by a collaborative research study with teacher-researchers who are using the results of Phase 1 on their own classes as part of extending the investigation to classroom/group settings. The aim is to examine how formative assessment can be used to improve attainment and attitudes, and finally how the work can be disseminated on a larger scale. Currently we have had the first year of the survey and the exploratory year of the collaborative study.

In Phase 1, comparison with the Concepts in Secondary Mathematics and Science (CSMS) study (Hart et al. 1981) will also enable us to examine how students' understandings have changed since 1976. The survey consists of three of the CSMS tests, Algebra, Ratio and Decimals, and an attitudes questionnaire. Some selected items from the original CSMS Fractions test were appended to the Ratio test. Results discussed here relate mainly to the Ratio and part of the Decimals tests.

In late June or early July 2008, tests were administered to a sample of approximately 3000 students from approximately 90 classes in 11 schools as shown in Table 1. Each student took two of the three tests so as to provide comparative information between tests but not to overload students. The numbers of students in each year-group taking each test is therefore roughly two-thirds of the total number of students involved in that age group in 2008.

Test	Year 7	Year 8	Year 9	Total
Ratio(+fractions)	680	754	588	2022
Decimals	717	767	598	2082

Table 1: Numbers taking the two test in 2008 sample

The sample of 20 schools involved in the survey across 2008 and 2009 is a stratified random sample drawn from MidYIS, the Middle Years Information System. MidYIS is a value added reporting system provided by Durham University, which is widely used across England (Tymms and Coe, 2003). Because some schools were not able to complete the testing in 2008, the 2008 sample for which results will be reported on an interim basis is not fully representative and is slightly skewed towards higher attaining students. The original CSMS test on Ratio was administered to a sample only from Years 8-10, although Decimals were used with Years 7-10 and Fractions with Years 7-9. In the current study, because the focus was on Key Stage 3, all the tests were administered to students in Years 7-9.

Ratio items, some Decimals items, and almost all of the small set of Fractions items used in this study, were designed to assess whether students could apply multiplicative reasoning in increasingly complex situations.

In the original CSMS data analysis in the 1970s, items were selected from each test to form a series of hierarchical levels of difficulty. These items included 20 out of the 27 items in the Ratio test and 39 out of 72 items in the Decimals test. The criteria for selected items and levels were consistent performance across different age-groups in the sample, high levels of correlation within the items for each level and strong hierarchical relationships between items in different levels (Hart et al. 1981). The remaining items that were retained were for diagnostic purposes. There were different numbers of levels associated with each test, since the levels were derived empirically; Ratio had 4 levels and Decimals had 6. The additional 15 items on fractions which were added to the Ratio test to better assess multiplicative reasoning, were drawn from items which appeared in the hierarchy of the CSMS Fractions test. However in the case of Fractions only single item comparisons with the 1976 data will be possible since the full set of Fractions items was not used.

Students were judged to have been successful at a specific level if they had successfully answered two-thirds or more of the items at that level. Students who had not achieved two-thirds of Level 1 items were said to be 'at Level 0'. It was possible to broadly describe the type of mathematical understanding required for the items in each level in each topic, although these were not always neat descriptions since the items and levels were assigned on empirical not theoretical grounds.

As part of the ICCAMS study, the CSMS tests were checked against the current version of the National Curriculum and appeared to form just as appropriate assessments as they had done in the 1970s. Piloting indicated that only minor updating of language or context for a very small number of items was required which would be unlikely to significantly affect their difficulty.

Phase 2 represents the intervention part of the project in which the aim is to design, trial and disseminate an intervention based on formative assessment practices in Year 8 in the areas of algebra and multiplicative reasoning. Two Year 8 teachers from each of four schools are part of the research team, with the schools having a range of intakes and attainment. The classes involved in 2008/9 covered almost the full range of Year 8 attainment. The work in the first year has been partly exploratory, with all members of the research team undertaking several school-based activities and reporting back in team discussions. These activities have included observing student responses in lessons relating to algebra and multiplicative reasoning, interviewing small groups of Year 8 students about aspects of their mathematical understanding, and trying out starters and other class activities as diagnostic tools.

Results and discussion

Because the full analysis of CSMS test results on the complete representative sample including those tested in Summer 2009 is not yet available, this account of Phase 1 results must be interpreted with some caution as it may differ somewhat from the final outcomes, especially because the mean MidYis score of the 2008 sample was a little higher than the national average.

The proportional bar-charts for the ratio results for Years 8 and 9 are shown in Figures 2 and 3 respectively. These suggest that in both year groups, as in other tests, there is a higher proportion below Level 1 in 2008 than in 1976. The reasons for this general finding are not yet clear. At the top levels the picture is more encouraging, although the improvements over 1976 are small enough to possibly disappear when the sample is larger and better balanced.

The verbal description of the levels is:

Level 1: Simple integer rates like ×2 or ×3 but including halving

Level 2: Rates like ×1.5 set in contexts that allow a 'building up' approach such as taking an amount then half as much again

Level 3: Problems involving fractional quantities or rates such as $\times 1^2/3$ set in contexts that allow an indirect 'building up' approach or which prompt the unitary method

Level 4: Non integer enlargement (e.g. 5:3) or problems set in a context involving scaling continuous quantities where a 'building up' approach (or the unitary method) is not meaningful

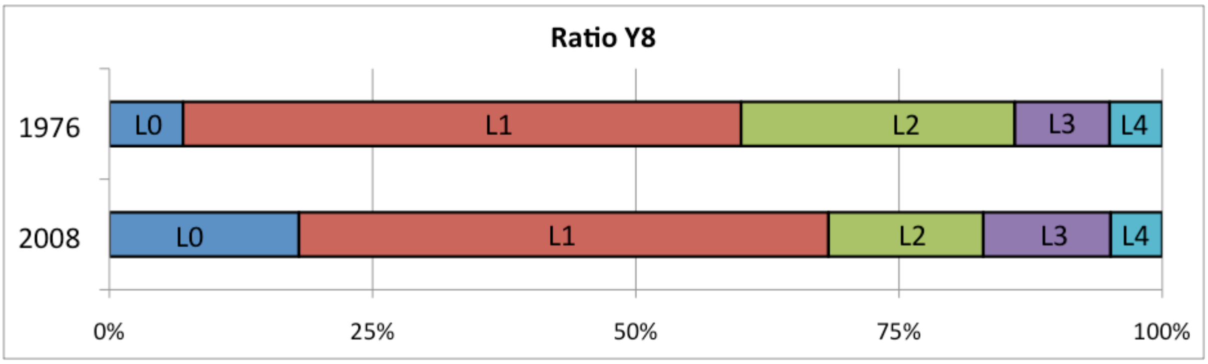

Figure 2: Proportions of Year 8 students assessed at the different CSMS levels for Ratio in 1976 and 2008 [N= 754 (2008), 800 (1976)]

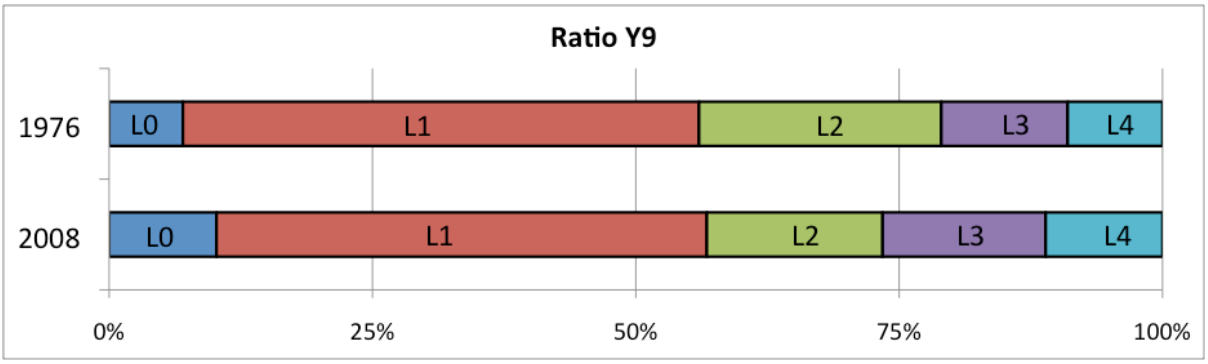

Figure 3: Proportions of Year 9 students assessed at the different CSMS levels for Ratio in 1976 and 2008 [N= 588 (2008), 767 (1976)]

So the results in Figures 2 and 3 suggest that two-thirds of Year 8 students and well over half Year 9 students cannot consistently manage anything in terms of rate and ratio other than reasoning involving whole number multiplication and division by 2, with about 20% at Year 8 and about 10% at Year 8 not even managing this reliably.

Currently it looks as though over 15% at Year 8 and over 25% at Year 9 can manage non-unitary ratio where some intermediate step is reasonably easy to spot. While only about 5% of Year 8 and 10% of Year 9 are fully operational with ratio, this proportion is equivalent at Year 8 and better at Year 9 than results for 1976.

Comparing these levels with the national strategies framework for Key Stages 3 and 4, there is not a direct match but it appears that in Years 7 and 8 students are expected to be taught the content at Level 3, and Level 4 is intended to be covered by the end of Year 9. (Some aspects of enlargement only appear in Year 10 but the examples given for Year 9 seem to suggest that calculation and use of scale factors like 0.7 is to be covered there.) The results therefore suggest that under 20% of students are succeeding on this material.

The proportions at different levels for the Decimals test were generally better than in 1977, except at the highest and lowest levels in Year 9. Most of the levels were defined in relation to understanding of the decimal place value system which arise in the measurement aspect of decimals, but part of the highest level is most closely related to the idea of multiplicative reasoning:

> *Level 6:* Decimals which result from division, and the existence of an infinite number of decimals

The proportion who reached Level 6 was about 10% at Year 8 and 15% at Year 9. The Year 8 result was better than in 1977 and the Year 9 result was worse, but again the margins were small and are thus unreliable. This result ties in well with the results for ratio as the understanding that you can obtain a decimal answer from a whole number division (and to find it with an easy divisor, such as 20) is needed for the highest level, Level 4, of the Ratio test. Again this Level 6 content is covered in the national strategy framework by the end of Year 8, but by the end of Year 9 still only 15% seem to have grasped it. (No comment is possible on Fractions levels as only a small number of 15 items were included in the ratio test.)

Looking at the proportions of items in each test on which there is significant improvement, significant reduction, or no significant change (Figure 4), it becomes clear that while there are improvements on over 50% of Decimal items, there are hardly any items with improvements on the Ratio test and none from the Fractions test.

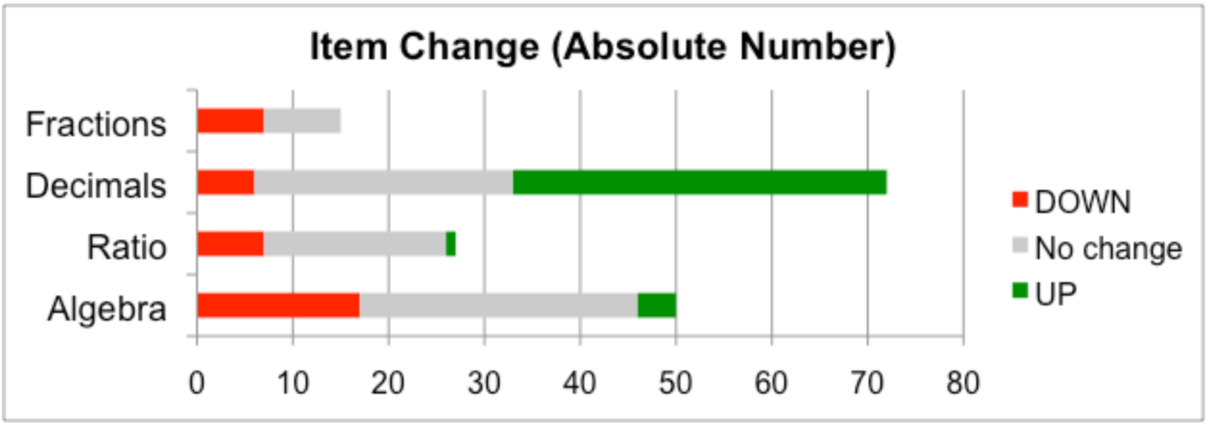

Figure 4: Numbers of items on which success rates have significantly increased or decreased between 1976/7 and 2008 in each mathematical topic using Year 9 data

Again many of the Decimals items with significantly improved scores were related to Decimals used for measurement rather than as operators. This leaves us therefore with the result that multiplicative reasoning seems to be generally weak and if anything to have deteriorated since the 1970s.

This is illustrated in Figure 5 by detailed results for one of the Level 4 items in the Ratio test. Although the ratio is an easy one of 2:3, the fact that the problem cannot be readily solved by the unitary method or by any form of addition strategy makes it difficult. Although enlargement first appears in the Year 8 strategy framework, and with non-whole number factors in Year 9, their answers indicate that many students at the end of Years 8 and 9 do not seem to appreciate that enlargement requires multiplicative rather than additive reasoning.

The results for both parts of the question are similar, as are the results for 1976 and 2008; the fact that the small initial lag in the 2008 results at the start of Year 9 closes by the end may be due to the inclusion of non-whole number enlargement in the Year 9 framework. However, even at the end of Year 9, only 20% of students can solve the problems.

To illustrate student thinking a summary is given of discussions on this ‘Kurly Ks’ item with two groups of Year 8 students, both from London comprehensive schools with GCSE results well above the national average.

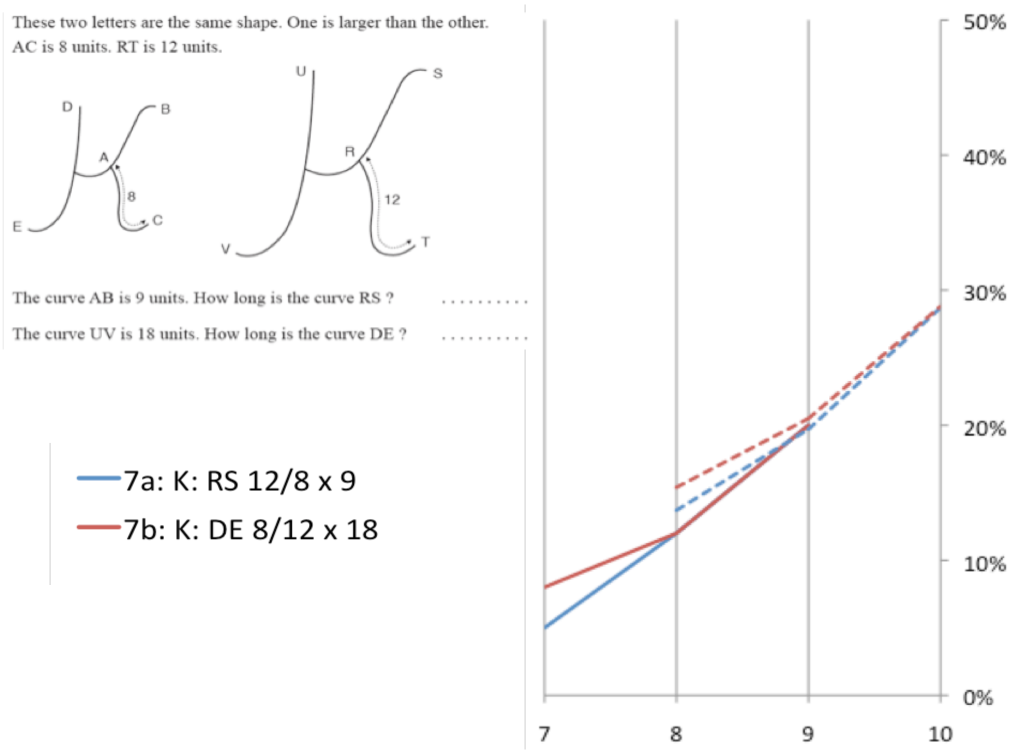

Figure 5: Success rates for Years 7 to 10 on an enlargement item in 1976 (dotted lines) and 2008 (unbroken lines)

Dietmar and Margaret with Nasreen, Ahmed, Maniyan and Susan (middle set) 28/11/08 Ahmed quickly launches in with ‘13 units… 4 more there... cos that’s 9, that will be 4 more the same...’. Others agree… They all went for an addition strategy, either 9+4 or 12+, to get 13. Dietmar then suggests: ‘Say instead of this being 12 we made it 16... and this bit’s still 9...’. Nasreen sticks to the addition strategy: ‘You add 8 now’. Ahmed agrees. Margaret suggests that instead of adding 8 you could double both numbers. Three of them say no, you have to add 8. (Not sure about Maniyan, whose English is weak.) Margaret asks what the answer would be if you did double the 9. Ahmed: ’18, and it’s supposed to be 17’ (Susan agrees, and Nasreen I think). They seem to regard doubling as a special case Ahmed: ’you don’t just do times 2 because it works on one answer’. Later Susan adds ‘You’re going to have to double if it’s twice as big, but if it is just under twice as big you can’t double everything.’ When Dietmar then sketched similar rectangles (8 by 9 and 16 by ?) they tended to prefer ‘doubling’ i.e. preferred the answer 18 to 17.

Dietmar and Jeremy with Bethan, Zack and Danny (top set) 10/11/08 Starts with a long silence...little whispers...Bethan: ‘8 is kinda equal to 12, in the same way that 9 is equal to RS’... pause…Dietmar: next step? Bethan says she is still thinking... Zack: ‘don’t really get... I was thinking... if the 8 here (points) and that’s 9, you plus it to find out that? but...’ [I now think he is working just on the little K and wants to add the 8 and 9 to find DE, but then thinks it might not work as the lines are curved – but at the time we didn’t appreciate this] He returns to RS on large K and now suggests 13. Bethan says she’s still not sure.... Jeremy to Zack: why 13? Zack: difference is 4... larger by 4... so this (RS) should be larger (than the 9) by 4.. if it’s an accurate enlargement. Bethan: I think it’s 13 and a half... cos 8 up to 12 is two-thirds... so 9 is two-thirds.. so I halved 9 which is 4.5 then add it onto 9 so you get . Jeremy: you think 13, you think 13.5... asks Bethan to explain again. She now uses multiplication rather than adding in the final step : ‘you halve it, then times it by 3 to get 3 thirds’. Zack later sticks to his additive strategy: like 8 and 12... the difference 4, hey... if you enlarge the smaller shape to get the bigger shape, wouldn’t everything have to have the same... difference? Bethan: I’m not sure if you have to add something or times it by something...

These interview quotations confirm that students from the middle of the attainment range, and also some from nearer the top, tend to use an addition strategy for enlargement. They reluctantly allow some exceptions, for example doubling, but do not see that they could use a multiplicative factor other than a whole number (Susan’s final comment is revealing). It is only a few higher attaining students, like Bethan, who are able to even if tentatively see their way through to multiplication strategies, even when easy scale factors as here allow ‘building up’ strategies.

Conclusions

Multiplicative reasoning is clearly an area which is key to both a large number of mathematical applications and to further study in mathematics. Yet the evidence presented here suggests that students have a very weak grasp of important aspects of the conceptual field and that that understanding in this area has not improved since the 1970s. This is not because these things are not taught as most of the students in the sample will have experienced teaching of the relevant ideas. We have to conclude that the teaching has not been very successful, and it is possible to speculate many reasons why this should be so. For example it may not have been appropriately related to students’ prior understandings, or may not have been sustained enough and broad enough to have had a permanent effect. In Phase 2 of the project we will be developing an intervention based on formative assessment and more sustained periods of teaching to see if this improves the learning of these important ideas.

References

Confrey, J., A. P. Maloney, K. H. Nguyen, G. Mojica, and M. Myers. 2009. Equipartitioning / splitting as a foundation of rational number reasoning using learning trajectories. In *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, edited by M. Tzekaki, M. Kaldrimidou and H. Sakonidis. Thessaloniki, Greece: PME.

Dickson, L., M. Brown, and O. Gibson. 1984. *Children learning mathematics: A teacher's guide to recent research*. Eastbourne: Holt, Rinehart & Winston Ltd.

Harel, G., and J. Confrey, eds. 1994. *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany, NY: SUNY.

Hart, K., M. L. Brown, D. E. Küchemann, D. Kerslake, G. Ruddock, and M. McCartney, eds. 1981. *Children's understanding of mathematics: 11-16*. London: John Murray.

Nunes, T., and P. Bryant. 2009. Paper 3: Understanding rational numbers and intensive quantities. In *Key understandings in mathematics learning*, edited by T. Nunes, P. Bryant and A. Watson. London: Nuffield Foundation. Available from www.nuffieldfoundation.org, accessed 9 October 2009.

Tymms, P., and R. Coe. 2003. Celebration of the Success of Distributed Research with Schools: the CEM Centre, Durham. *British Educational Research Journal* 29 (5):639-653.