Multiplicative Reasoning: Lesson 2B

Westgate Close Revisited

These two lines show distances in metres and feet along Westgate Close.

Eric wants to convert 66 metres into feet.
He numbers the lines so he can read-off the answer.

metres
0   15

0   50
feet

Summary

In this lesson, students consolidate ideas from Lesson 2A. They carefully number a double number line (DNL) for converting metres to feet (and feet to metres). They then create similar conversion devices using a mapping diagram and a Cartesian graph, and consider how multiplication is modelled by these different representations.

Outline of the lesson

1. Discuss the above conversion task.

   - Discuss different ways of converting 66 m to ft.
   - Discuss Eric's method: How does he number the lines?
     Where is 66 m? What is the number underneath this?

2. Students number a DNL to represent metres and feet.

   - Distribute the MR-2B Worksheet (see page 5). Point to the DNL.
   - Ask students to number the blue marks (only).
   - Discuss the numbers represented by the marks (blue and grey).
     How are the numbers spaced-out on each line?
     How big are the 'gaps' between adjacent marks?
     What is the relation between vertically aligned numbers?
   - Use the numbered DNL to go over Eric's method.

3. Use the numbered double number line.

   - Locate the postions on the DNL of the numbers from Tasks B and C
     of Lesson 2A [ie 15 and 50, 72 and ?; 6 and 20, 33 and ?].
   - Discuss some other metres-feet conversions (of your or the students'
     choosing). Which numbers are easy? Which are more difficult?

4. Draw a mapping diagram for metres and feet.

   - Use the mapping diagram on the worksheet. Draw arrows to
     represent (some of) the number-pairs from Stage 3 of the lesson.
   - Discuss the pattern made by the arrows - their slope, the space
     between them, whether they meet.

5. Draw a Cartesian graph for metres and feet.

   - Use the Cartesian axes on the worksheet.
     Plot points for (some of) the previously considered number-pairs.
   - Discuss the pattern made by the dots.

6. Compare the three representations.

   - How do various features of the representations correspond?
   - What are the strengths of the different representations?
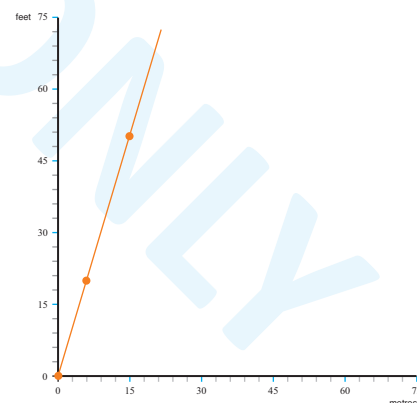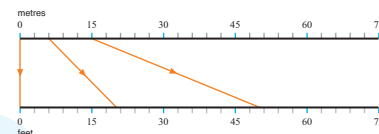
Overview

Mathematical ideas

In this lesson, students revisit the Westgate Close problem in order to consider the double number line (DNL) in more detail. Students focus on the linear nature of the number line scales and then compare the DNL with a mapping diagram and a Cartesian graph. All three representations provide models for thinking about multiplication (and for countering the 'addition strategy'). Also, the DNL and Cartesian graph provide instant 'ready reckoners' for reading-off conversions (of metres into feet), albeit reckoners that may not be very precise in practice.

Students' mathematical experiences

Students

- scrutinise linear scales

- use different representations to model a multiplicative situation

- compare representations.

Key questions

In what ways are the diagrams the same?

In what ways are the diagrams different?

Assessment and feedback

Do students realise that the DNL scales are both linear but numbered differently? Can they use the Westgate Close context to explain why?

Observe which, if any, of the diagrams help students see that the metres-feet conversion tasks are multiplicative rather than additive. Do they help students who used the 'addition strategy' on the Starter mini test?

Encourage students to describe the various representations in detail, and to read them 'dynamically'. What happens to the values (in metres and feet) as one moves steadily along the double number line (or along the straight line graph)? What changes, and what stays the same?

Adapting the lesson

Consider other conversions in everyday life or in science. For example, a ruler marked in cm and inches, or a measuring cylinder marked in pints and litres (what happens to the scales if the container tapers as with a measuring jug?). Look at some conversion charts and tables. [Note: in Lesson MR-4A we look at a currency conversion.]

What about temperature in ˚C and ˚F ? Here the 0s don't line up [See page 4].

You might want discuss how the three representations can be used to model the items in the *Mini Ratio Test*. How can the models help us refute the 'addition strategy'?

Outline of the lesson (annotated)

1. Discuss the above conversion task.

    - Discuss different ways of converting 66 m to ft.
    - Discuss Eric's method: How does he number the lines? Where is 66 m? What is the number underneath this?

    —— We can 'skip' along the lines:
    15,50 to 30,100 to 45,150 to 60,200 to 63,210 to 66,220.
    Or we could find 3,10 and multiply by 22, or find 6,20 and multiply by 11.
    Or we can find the multiplier × 3⅓ than maps 15 onto 50 and apply this to 66.

2. Students number a DNL to represent metres and feet.

    - Distribute the MR-2B Worksheet (see page 5). Point to the DNL.
    - Ask students to number the blue marks (only).

    —— This is to help students realise that the scales are linear but numbered differently: we are representing the fact that every 15 m inteval is equivalent to 50 ft.

    - Discuss the numbers represented by the marks (blue and grey).
    How are the numbers spaced-out on each line?
    How big are the 'gaps' between adjacent marks?
    What is the relation between vertically aligned numbers?
    - Use the numbered DNL to go over Eric's method.

    —— Allow plenty of time for students to contemplate what is goin on, here and in later stages of the lesson. Students might notice that
    - the numbers are evenly spaced (the scales are linear).
    - the gaps represent 3 m and 10 ft.
    - number of metres × 3⅓ = number of feet, or $x \rightarrow 3\frac{1}{3}x$.

3. Use the numbered double number line.

    - Locate the postions on the DNL of the numbers from Tasks B and C of Lesson 2A
    [ie 15 and 50, 72 and ?; 6 and 20, 33 and ?].

    —— It is relatively straightforward to locate the 6 m mark and the coresponding 20 ft mark (for Task C), and to locate the 72 m and 33 m marks, and hence to estimate the corresponding values in feet (240 ft and 110 ft respectively).

    - Discuss some other metres-feet conversions (of your or the students' choosing). Which numbers are easy? Which are more difficult?

    You might want to challenge students to read-off conversions where neither value lies on a given mark. How could one use a calculator to make the conversions?

4. Draw a mapping diagram for metres and feet.

    - Use the mapping diagram on the worksheet. Draw arrows to represent (some of) the number-pairs from Stage 3 of the lesson.

    —— It is not easy to draw accurate arrows for the given scales and markings. However, student should notice that the arrows splay out and get flatter and flatter.

    - Discuss the pattern made by the arrows - their slope, the space between them, whether they meet.

    We have drawn the two number lines 2⅓ cm apart. It turns out that the arrows meet at a point 1 cm above the zero mark of the top number line. Why?!

5. Draw a Cartesian graph for metres and feet.

    - Use the Cartesian axes on the worksheet. Plot points for (some of) the previously considered number-pairs.
    - Discuss the pattern made by the dots.

6. Compare the three representations.

    - How do various features of the representations correspond?
    - What are the strengths of the different representations?

    —— We have considered three diagrammatic ways to represent the scaling relationship of × 3⅓ or $x \rightarrow 3\frac{1}{3}x$. On the DNL any number on the bottom scale is 3⅓ times the number above it; on the mapping diagram any two arrows are 3⅓ times as far apart at the head than the foot; on the Cartesian graph, the straight line has a gradient of 3⅓.

    The scaling maps 0 → 0. On the DNL the zeros line up; on the mapping diagram the zeros are joined by an arrow; the Cartesian graph goes through the origin.

    A strength of a mapping diagram is that one can choose any practicable scale (as long as it is linear and there is room on the page to record all the desired values). In contrast to the DNL, one does not have to work out the numbering system for the second scale, as it is the same as the first. However, one can't simply read-off conversions as one can with the DNL and the Cartesian graph. And for the latter one can use any two linear scales, and (in theory) just one plotted point, eg (15, 50), joined with a straight line to the origin.

Background

The double number line (DNL)

The DNL offers a neat way of representing multiplicative relations. We see it as a very useful model for *thinking* about multiplication (though it is sometimes less useful for actually *solving* multiplication problems).

Consider the pair of lines A and B (first diagram, below), where 0 on line A is lined-up with 0 on line B and where 5 on A is lined up with 8 on B. We know that 8 is 1.6 times 5.

If we mark linear scales on each line, then any number on scale B will be 1.6 times the corresponding (lined-up) number on scale A (second diagram, below).

(In practice, drawing such linear scales accurately can be quite a challenge, and we would not always want students to do this - rather, we want them to appreciate the *idea* that the scales illustrate.)
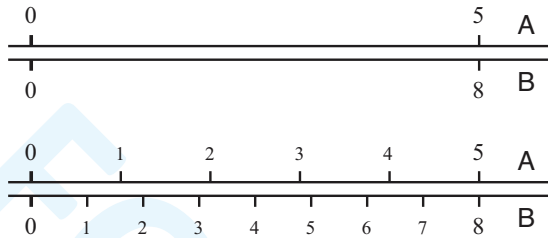
Scales on maps are well known examples of a double number line. There are two sorts. In one we are shown how distances on the actual map (measured in cm, say) correspond to distances on the object depicted by the map (measured in km, say). Such a scale is shown below, on a ruler used by model railway enthusiasts.
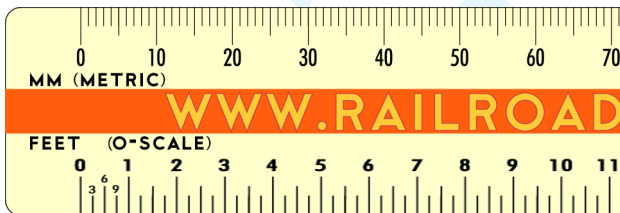
Scales of this sort are often also expressed numerically, as the ratio of a distance on the map (or ruler) to the corresponding distance in real life. In the case of the 0-scale used for model railways, this is commonly 1:43.5.

The other kind of map scale shows how distances represented on the map can be read in different units (for example, feet and metres). The example, right, is from Google Maps.
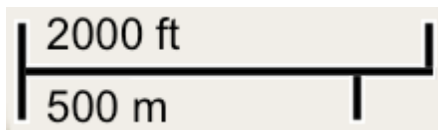
In effect, this kind of scale converts one unit to another. We can use such scales to represent other conversions, not just involving distance. For example, if we are told that 8.5 hectares is equivalent to 21 acres, we can construct a conversion scale as follows:

Draw two parallel lines to represent hectares and acres, and line-up 0 with 0 and 8.5 with 21(first diagram, below).

Draw linear scales (second diagram, below).

However, we have to be careful. Say we know that 10°C is the same as 50°F; we can't create a conversion scale by simply lining up 0 with 0 and 10 with 50 (first diagram, below). The equivalent of 0°C is 32°F, not 0°F (as illustrated in the second diagram, below). Does this diagram work?
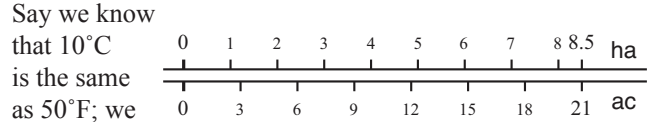
In each of these two examples, the scales on the two number lines have been different. We could keep the scales the same, but we would then have to show how the numbers on the scales correspond, for example by using arrows. Such a diagram is usually called a mapping diagram. The diagram below is for our first example, ie for the × 1.6 mapping.
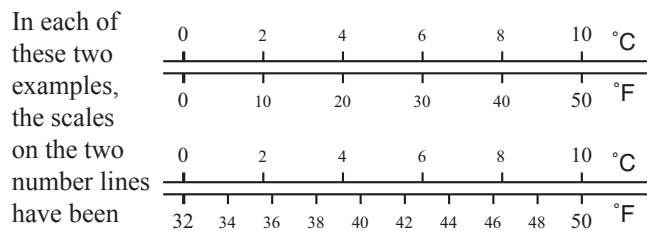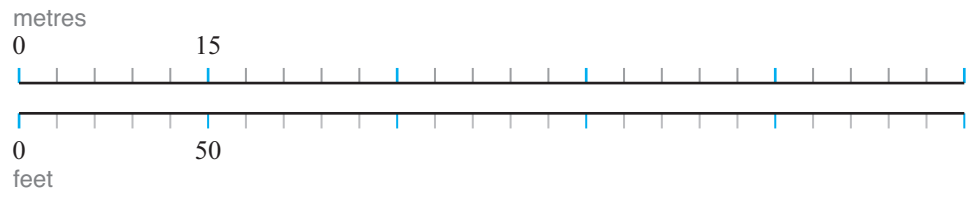
If we rotate one of the number lines through 90°, we can change the mapping diagram into a Cartesian graph, as in the two steps shown below.
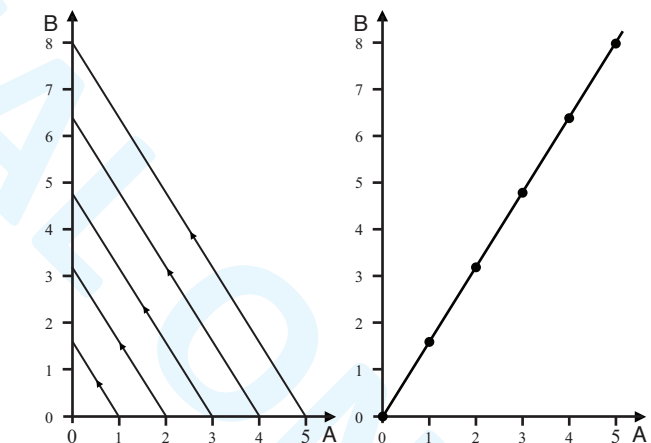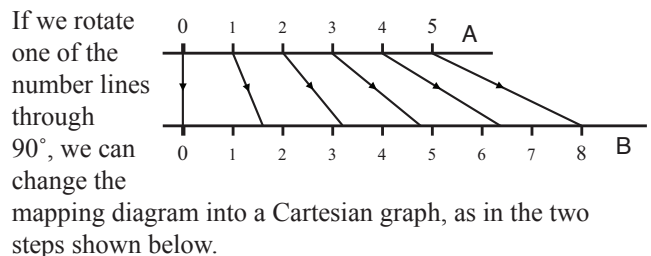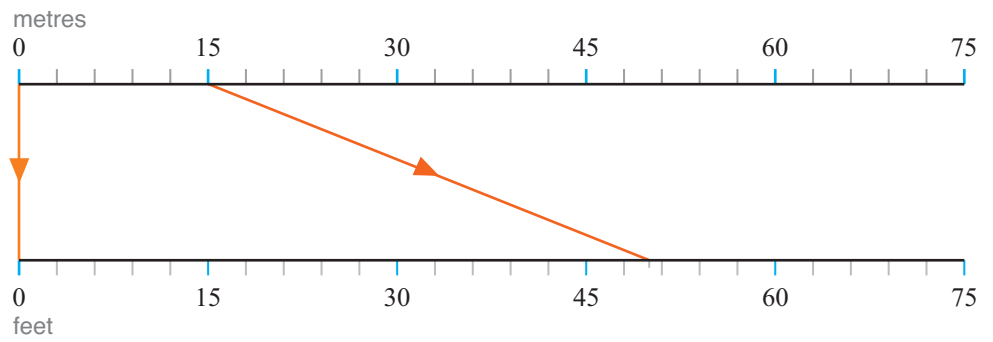
[We can of course go straight to Cartesian graphs if we want a device for reading-off conversions. Here we can draw the linear scales on the axes first, using any scale that suits the paper and ruler we are using (conceptually, there is a lot to be said for keeping the scales on the two axes the same). Then we usually only need one pair of values (eg £52 buys $70) which we plot as a point and then join to the origin (usually!) with a straight line (usually!).]

MR-2B Worksheet

Double number line

metres
0   15

0   50
feet

Mapping diagram

metres
0   15   30   45   60   75

0   15   30   45   60   75
feet

Cartesian graph

feet 75
60
45
30
15
0
     0   15   30   45   60   75
                                            metres