

Chapter XX

Learning Experiences Designed to Develop Algebraic Thinking: Lessons from the ICCAMS Project in England

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Algebra provides powerful tools for expressing relationships and investigating mathematical structure. It is key to success in mathematics, science, engineering and other numerate disciplines beyond school as well as in the workplace. Yet many learners do not appreciate the power and value of algebra, seeing it as a system of arbitrary rules. This may be because teaching often emphasises the procedural manipulation of symbols over a more conceptual understanding. In this chapter, we will draw on our experiences from the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICAMS) study in order to look at ways in which learning experiences can be planned. In doing so, we will discuss how representations can be used, and the relationship between algebra and other mathematical ideas strengthened. We will also discuss how formative assessment can be used to nurture a more conceptual and reflective understanding of mathematics.

1 Introduction

Why should I learn algebra, I don't want to be a maths teacher. (A middle attaining 13 year old in England)

Algebra is a central topic within the school mathematics curriculum because of its power both within mathematics and beyond. Algebra can be used to model and predict and is thus key to science, engineering, health, economics and many other disciplines in higher education and in the workplace (Hodgen & Marks, 2013). Yet, too often we fail to communicate this power to learners who, like the 13 year old above, perceive algebra to be something that is only useful in school mathematics lessons.

The research evidence on participation in mathematics indicates that the main obstacle lies in negative learner attitudes (e.g. Matthews & Pepper, 2007; Brown, Brown, & Bibby, 2008). Most learners do not want to carry on with their mathematical studies because they believe they are not ‘good at mathematics’, and ‘did not understand it’. They also found it ‘boring’ and ‘unrelated to real life’. These negative attitudes apply even to many high attaining learners. Mendick (2006), for example, quotes a high attaining learner studying advanced mathematics:

What’s the use of maths? ... when you graduate or when you get a job, nobody’s gonna come into your office and tell you: ‘Can [you] solve x square minus you know?’ ... It really doesn’t make sense to me. I mean it’s good we’re doing it. It helps you to like crack your brain, think more and you know, and all those things. But like, nobody comes [to] see you and say ‘can [you] solve this?’

One can, of course, point to many contexts in which quadratics do prove useful as Budd and Sangwin (2004) have done. But we should also consider whether what we do in our mathematics classrooms could be contributing to this problem. Do we consider the difficulties that learners have with algebra sufficiently? Do we focus too much attention on algebraic manipulation and the ‘rules’ of algebra? Could we teach algebra in a way that conveyed its power to all learners?

In this chapter, we discuss how we addressed these problems in the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) project. In doing so, we consider the difficulties learners face when understanding algebra.

2 Background

ICCAMS was a 4½ year project funded by the Economic and Social Research Council in the UK. Phase 1 consisted of a survey of 11-14 years olds' understandings of algebra and multiplicative reasoning, and their attitudes to mathematics (Hodgen et al., 2010). Phase 2 was a collaborative research study with a group of teachers which aimed to improve learners' attainment and attitudes in these two areas (Brown, Hodgen, & Küchemann, 2012). Phase 3 involved a larger scale trial with a wider group of teachers and students. ICCAMS was funded as part of a wider initiative¹ aimed at increasing participation in STEM subjects in the later years of secondary school and university, a concern shared by many countries around the world including Singapore.

The Phase 1 ICCAMS survey involved a test of algebra first used in 1976 in the seminal Concepts in Secondary Mathematics and Science (CSMS) study (Hart & Johnson, 1983; Hart et al., 1982). In 2008 and 2009, the algebra test was administered to a sample of around 5000 learners aged 12-14 from schools randomly chosen to represent learners in England.

The CSMS algebra test was carefully designed over the 5-year project starting with diagnostic interviews. The original test consisted of 51 items.² Of these 51 items, 30 were found to perform consistently across the sample and were reported in the form of a hierarchy (Booth, 1981; Küchemann, 1981). Piloting indicated that only minor updating of language and contexts was required for the 2008/9 administration.

By using the same test that was used in the 1970s, we were able to compare how algebraic understanding had changed over the 30-year interval. Over the intervening period, there have been several large scale national initiatives that have attempted to improve mathematics teaching and learning, including learners' understanding of algebra (for a discussion of these initiatives, see Brown, 2011; Brown & Hodgen, 2013). Hence, it was a serious concern that the comparison showed that learners' understanding of algebra had fallen over time (Hodgen et al., 2010). It was in this context that we designed an approach to teaching that was intended to address learners' difficulties. However, before doing

so, it is important to set out clearly exactly what we mean by ‘algebraic understanding’.

2.1 What is algebraic understanding?

The CSMS test aims to test algebraic understanding by using “problems which were recognisably connected to the mathematics curriculum but which would require the child to use methods which were not obviously ‘rules’.” (Hart & Johnson, 1983, p.2). The test items range from the basic to the sophisticated allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of learner understanding. The focus of the test was on generalised arithmetic. Items were devised to bring out these six categories (Küchemann, 1981):

Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable.

Item 5c presented the following problem to learners:

If $e + f = 8$, $e + f + g = \dots$

Here the letters e and f could be given a value or could even be ignored; however the letter g has to be treated as at least a specific unknown which is operated upon: the item was designed to test whether learners would readily ‘accept the lack of closure’ (Collis, 1972) of the expression $8 + g$. Learners tend to see the expression as an instruction to do something and many are reluctant to accept that it can also be seen as an entity (in this case, a number) in its own right (Sfard, 1989). Thus, of the learners aged 13-14 tested in 1976, only 41% gave the response $8 + g$ (another 34% gave the values 12, 9 or 15 for $e + f + g$, and 3% wrote $8g$).

In question 13, learners were asked to simplify various expressions in a and b . Some of the items could also readily be solved by interpreting the letters as objects, be it as as and bs in their own right, or as a shorthand for apples and bananas, say (eg 13a: simplify $2a + 5a$; 13d: simplify $2a + 5b + a$); however, such interpretations become strained for

an item like 13h (simplify $3a - b + a$), where it is difficult to make sense of subtracting a b (or a banana).

3 Current Approaches to Teaching in England

School algebra for 11-16 year olds in England focuses on the use of letters as *specific unknowns* rather than variables (Küchemann, Hodgen, & Brown, 2011a).³ Also, if one looks at the more common school textbooks, the algebra is often not about anything, or at least not about anything meaningful (Hodgen, Küchemann, & Brown, 2010). Consider the example reproduced in Figure 1, which is on a page headed ‘Solving problems with equations’ from a homework book for learners aged 12-13.

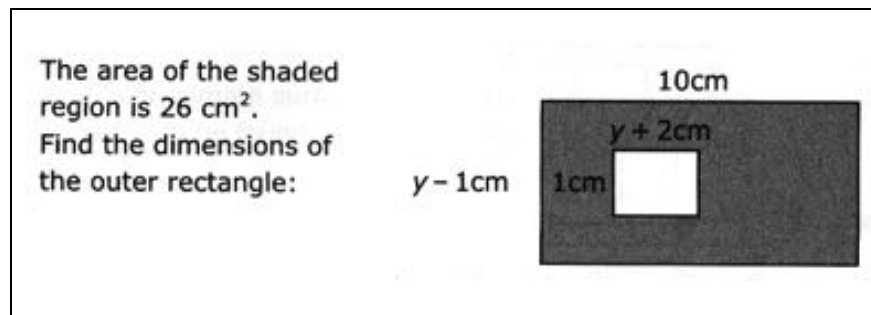


Figure 1. An example from a typical English lower secondary mathematics textbook

Here we are expected to construct and solve an equation to find a specific value of y and then to use this to find the dimensions of the specific rectangle that fits the given conditions. But under what circumstances (apart from when asked to practise algebraic procedures) would we want to find such an answer? And out of what kind of situation would the given expressions $y - 1$ and $y + 2$ come about? The problem becomes a lot more engaging, though not necessarily more credible, if we let y vary. What values of y ‘make sense’ here? What happens to the shapes of the rectangles? What is the relation between the height of the

grey rectangle and the width of the white rectangle (and why have the book's authors not bothered to show this in the diagram)?

Learners most commonly meet the idea of letters as variables in the context of the Cartesian graph. Here the work is almost exclusively about straight line graphs and plotting the graph of a given function, or finding the function of a given graph. This is commonly done by focusing on the gradient and y -intercept on the graph, and equating this to m and c in the standard algebraic representation of the function.⁴ Here, too, the work is rarely about anything. Indeed, our interviews suggest that many learners do not realise that the work is about sets of points and, moreover, points whose coordinates fit a particular relation. This contrasts strongly with the approach advocated in the best-selling *School Mathematics Project* textbook for lower secondary in the 1970s (Hodgen, Küchemann, & Brown, 2010). Here, emphasis is placed on the graph as representing a set of points all of which satisfy the relation by a consideration of the graphs with “more and more” intermediate values (see Figure 2). After “join[ing] up the points by drawing a line with a ruler”, the learner is asked to consider whether the point $(0, 0)$ lies on the line and “Is it true that for every point on the line, the second coordinate is always three times the first coordinate?” (p.93).

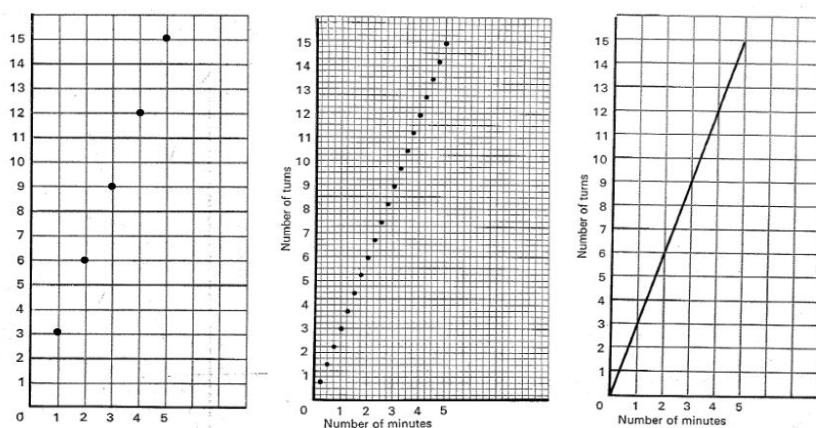


Figure 2. Three figures from a 1970s textbook

4 Our Design Principles

To counter these shortcomings, we developed a set of design principles. A key concern was to develop algebra lessons which had at least some kind of a ‘realistic’ context. We note that by ‘realistic’, we do not mean real life contexts that the learners may have encountered, but rather contexts that the learners can imagine (and indeed, in some cases, this involved a ‘pure’ context). In doing so, we aimed to design task that were intriguing and which provided opportunities for what Streefland (1991) refers to as the ‘insightful construction of structures’ (p.19). We also aimed to bring together the often fragmented activities of tabulating values, solving equations, drawing graphs, and forming and transforming algebraic expressions and relations. In addition, we drew on approaches for which there is research evidence to indicate they are effective in raising attainment (Brown et al., 2012). These included formative assessment (e.g., Black & Wiliam, 1998), connectionist teaching (e.g., Askew et al, 1997; Swan, 2006), collaborative work (e.g., Slavin, Lake, & Groff, 2009; Hattie, 2009) and the use of multiple representations (e.g., Streefland, 1993; Gravemeijer, 1999; Swan, 2008). In particular, multiple representations, such as the Cartesian graph and the double number line (see, e.g., Küchemann, Hodgen, & Brown, 2011b), are used both to help learners better understand and connect mathematical ideas and to help teachers appreciate learners’ difficulties.

5 Nurturing Conceptual Understanding of Algebra

We discuss two approaches that we used, both of which link back to our earlier analysis of the teaching of algebra in England. The first takes a static textbook problem and attempts to introduce a more dynamic – and intriguing – element. The second examines how the Cartesian graph might be better introduced.

5.1 *A more dynamic approach to a textbook algebra problem*

In Figure 3, we show a task from a current English textbook, which, like the task described earlier, appears to provide little interest or intrigue.

Indeed, the triangle pictured appears to be isosceles and the height is quite visibly not twice its base.

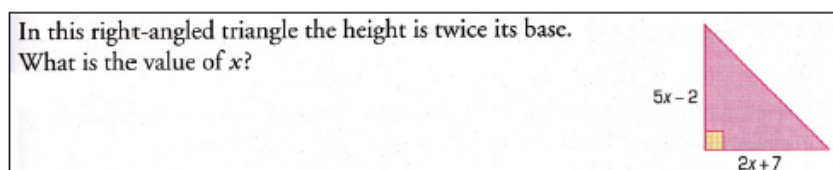


Figure 3. A static task from an English textbook

In Figure 4, we show the task as presented in an ICCAMS lesson. In one sense, the change to the task is very slight – the diagram is almost exactly the same. Yet, the additional question transforms the task to one where the learners have to think of x as a variable (or at least to consider different values for x) and then imagine what happens to the triangle as x changes. Indeed, one can start to consider whether the height *can* ever be twice the base. This might prompt further questions such as can the height ever be 10 times the base (and what would it mean for x to be negative? Is this allowed?).

Algebra: Lesson 6A

Growing triangle

1. What happens to the triangle as x changes?
2. For what value of x is the height of the triangle twice its base?

Figure 4. A more dynamic version of the task

It is a short step then to use dynamic geometry to support learners' imaginations (and to compare the original triangle to one where the height actually *is* twice the base, $4x+14$).⁵

5.2 Making connections: Variables, tables and the cartesian graph

One of the most interesting items on the CSMS algebra test posed the following question to learners, “Which is larger, $2n$ or $n+2$?” and asked them to explain their answer. Commonly, learners would opt for $2n$ and give a justification along the lines of ‘Because it’s multiply’. We were interested in whether learners would realise that the difference between the expressions varies and that there are values of n for which $2n$ can be smaller, the same, or larger than $n+2$. Very few learners demonstrated such an awareness and so we designed lessons that addressed this. One ICCAMS lesson sequence was modelled very closely on this item and began with a short whole class assessment task ‘Which is larger, $3n$ or $n+3$?’. This task was designed to enable the teacher to listen to the learners’ ideas and then consider (or diagnose) their (mis)-understandings prior to teaching a full lesson. (See Appendix for a reproduction of this task and the guidance given to the teacher on ‘diagnosis’.) Here the context was entirely ‘pure’ but we designed the main task of the lesson that immediately followed this assessment to involve a ‘real’ context - about hiring a boat.

Algebra: Lesson 1A

Boat Hire

Olaf is spending the day at a lake.
He wants to hire a rowing boat for some of the time.

Freya’s Boat Hire charges £5 per hour.
Polly’s Boat Hire charges £10 plus £1 per hour.

Whose boat should Olaf choose?

Figure 5. The Boat Hire problem

The *Boat hire* lesson started with the problem reproduced in Figure 5. The task is ‘realistic’ in the sense that learners can imagine such a scenario and think their way into it, even though they might never have

encountered such a problem in real life, and perhaps never will. Learners found the task engaging, because they could make sense of it and because initially they came up with different conclusions which had to be resolved.

After a brief period of discussion, as a class and in small groups, the teacher is asked to record on the board the numerical data that learners come up with to support their arguments. The data are first recorded 'randomly' and then (perhaps prompted by the learners themselves) in an ordered table. Learners are used to using ordered tables, but this gives them an opportunity to see why such an ordering can be helpful.

Figure 6 shows one pair of learners' work, who, having ordered the data started noticing that the differences (-6, -2, +2, +6) form a pattern. Such analysis can prompt the question 'Are the hire costs ever the same?'. One way we suggest of pursuing this is to represent the relations algebraically (eg $5a$ and $10 + a$), which might lead learners in some classes to consider how to solve the equation $5a = 10 + a$.

	P	F
1 hour	£11	£5
		-6
2 hours	£12	£10
		-2
3 hours	£13	£15
		+2
4 hours	£14	£20
		+6

Figure 6. Two learners tabular recording of the two expressions

We also suggest putting the data on a Cartesian graph (see Figure 7). Such a representation is quite abstract (the 'picture' isn't of boats on a

lake). However, because the graph is about a by-now familiar story, learners are in a good position to relate salient features of the graph to the story and the other representations. One such feature is the point where the two dotted lines cross; another might be the gradient of the lines (what does this tell us, and how is the same thing manifested in a table or algebraic expression?); or the point where a line crosses the y -axis (or, indeed, the x -axis!); or can lines meaningfully be drawn through the points (what do the intermediate points represent, and do the resulting points satisfy the relation in the table or algebraic expressions?).

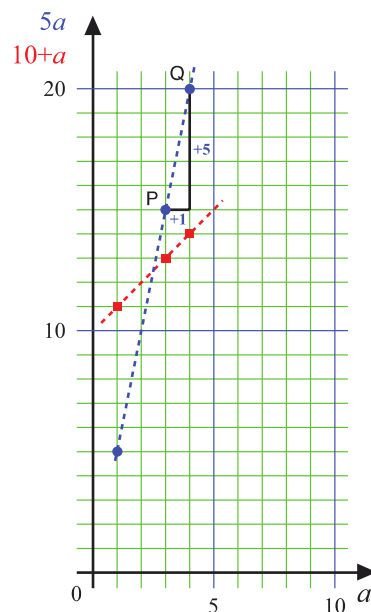


Figure 7. Using a Cartesian graph to represent $5a$ and $10+a$

Here learners had the opportunity to see that a graph can be meaningful and useful and it sometimes lead learners to draw graphs spontaneously, eg to compare algebraic expressions. Of course, learners often had difficulties in drawing effective graphs, eg through not numbering the axes in uniform intervals, but this in itself could be a useful experience.

6 Conclusion

The ICCAMS lessons and approach was designed and trialled in the English context. In this context, the wider trial in Phase 3 of the ICCAMS study showed a significant effect; over a year the rate of learning for those learners who had experienced the lessons was double that of those who had not (Hodgen et al., 2014). We believe that the lessons, and the general approach, have wider value. Indeed, although many learners in Singapore are likely to have a better understanding of algebra than many learners in England (OECD, 2013), it is likely that they will benefit from this kind of experience.⁶ In particular, there is very good evidence to indicate that algebraic understanding can be developed through an approach based on intriguing problems, making connections, promoting collaborative work, using multiple representations and diagnosing learners' (mis)-understandings (e.g., Watson, 2009). The ICCAMS lessons provide one way of supporting teachers to do this.

Notes

1. ICCAMS was part of the Targeted Initiative on Science and Mathematics Education (TISME) programme. For further information, see: tisme-scienceandmaths.org/
2. The Algebra test is available for non-commercial purposes (research and teaching) by contacting the authors.
3. An English translation of this work (Küchemann et al., 2011a) is available from the authors.
4. In England, straight line graphs are commonly referred to as $y=mx+c$.
5. A GeoGebra file of this activity is available from the authors.
6. The ICCAMS lessons are available for trialling by interested teachers and schools by contacting:
jeremy.hodgen@kcl.ac.uk or dietmar.kuchemann@kcl.ac.uk.
See also: <http://iccams-maths.org>

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Appendix

Guidance for $3n$ or $n+3$ assessment activity

Algebra: Lesson 1 STARTER

Which is larger, $3n$ or $n + 3$?

Commentary

The aim of this starter is to see what approaches students use to compare algebraic expressions.

- Do students understand the algebraic notation?
- Do they focus on the operations ('multiplication makes bigger')?
- Do they evaluate the expressions for specific values of n ?
- Do they respond to the fact that we don't know the value of n ?
- Do they realise that the difference between the expressions might change as n varies?

Use the starter a few days before teaching the two lessons.